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**GÖTEBORGS UNIVERSITET**

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# Three Perspectives of Schiemann's Theorem

*Master's thesis in mathematics.*

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*Master's thesis for the master's program in mathematics at the university of  
Gothenburg*

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## Abstract

Interest in the field of spectral geometry, the study of how analytic and geometric properties of manifolds are related, was sparked when Marc Kac in 1966 asked the question “can one hear the shape of a drum?”. One of the problems that garnered attention because of this was whether the Laplace spectrum of a flat torus determines its shape, even though it was not new. The final answer to this question is due to Alexander Schiemann and it turns out to be yes if and only if the dimension of the flat torus is 3 or lower. His results are not widely known in today’s thriving spectral geometry community and there are two main reasons for this. Firstly, his published thesis and article are entirely number theoretical and never mention the related spectral geometry. Secondly, the thesis is written in german and the proof is quite technical.

The reason why the spectral geometry of flat tori is particularly interesting is its connection to the geometry of lattices and the number theory of positive definite forms over the integers. In this thesis we aim to present this subject and its different perspectives. We especially focus on the details of Schiemann’s proof that ternary positive definite forms are determined by their representation numbers over the integers. Building on his techniques, we finally discuss some open problems and ideas for how to solve them.

Keywords: Lattice theory, Minkowski reduction, flat torus, isospectrality, spectral determination, positive definite quadratic forms, Alexander Schiemann.

## Preface

I would first and foremost like to thank David Sjögren who was originally a part of this project. His main contributions are that he translated parts of Schiemann's PhD thesis and that he helped me with ideas for proofs of the second inheritance theorem. The help of my examiner Martin Raum has been invaluable. He has taken a lot of time to help me and explain what I needed to know about Julia and computer science in order to make the necessary computations in chapter 5. Finally, I want to thank my supervisor Julie Rowlett for interesting discussions and positive encouragements.

Felix Rydell, Gothenburg, June 2020.

# Nomenclature

<i>Symbol:</i>	<i>Explanation:</i>
$\  \cdot \ $	The Euclidean norm $\  \cdot \ _2$ on $\mathbb{R}^n$ .
$e_i$	The standard unit vectors.
$\mathbb{R}_0^+, \mathbb{N}_0, \mathbb{N}_1$	The set of all non-negative real numbers, the set of non-negative integers and the set of positive integers respectively.
$\mathcal{R}, \mathcal{R}_X$	Definition 5.0.1.
$GL_n(\mathbb{R}), GL_n(\mathbb{Z}), O_n(\mathbb{R})$	The standard matrix groups. I refer to the orthogonal group as the <i>orthonormal group</i> .
$[v_j]$	Column vector notation for a matrix.
$\Gamma^n$	The product $\Gamma \times \cdots \times \Gamma$ consisting of $n$ lattices.
$S_n$	The group of permutations of $\{1, \dots, n\}$ .
$\Gamma$ -periodic function	A function $g : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $g(x + \gamma) = g(x)$ for all $x \in \mathbb{R}^n$ and $\gamma \in \Gamma$ .
$N_n$	Definition 2.4.1.
$\text{diag}(d_i)$	A diagonal matrix with the elements $d_i$ along the diagonal.
$R_A$	The fundamental domain of a basis matrix $A$ . Definition 1.2.8.
$\Delta$	The Laplace operator. Defined in chapter 2.
$\mathcal{C}$	A relation between flat tori and lattices. Definition 1.2.6 and 2.0.2.
$\mathcal{I}$	A relation between flat tori and lattices. Definition 2.3.1 and 2.3.9.
$\delta_{ij}$	The function that takes the value 1 if $i = j$ and 0 otherwise.
$P_c$	A polyhedral cone. Definition 4.2.3.
$v^{\geq 0}, v^\perp$	Definition 4.4.2.
$\mathbb{Z}_*^3$	Definition 5.0.2.
$V, V_n$	Definition 5.1.2 and 6.3.1.
$D, D_n^k$	Definition 5.1.6 and 6.3.2.
$\dot{\cup}$	Disjoint union
$M, M_1, M_2, M_3$	Sets of edges of $\bar{V}$ where $M = M_1 \cup M_2 \cup M_3$ . Defined in section 5.1.
$M'_3$	Defined in the proof of proposition 5.1.5.
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$	Corresponds to the conditions of sign reduction. Defined in section 5.2.1.
$W_{0,a}, W_{1,a}, W_{2,a}$	Defined in section 5.4.
$\odot$	The premier choice for a torus-like Q.E.D symbol.

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# Chapter 0

## Introduction

One purpose of this thesis is to explain how the subject of spectral geometry of flat tori, the geometry of lattices and the number theory of positive definite forms are connected, and give an overview of known results. However the main goal is to highlight and do the details of Schiemann's somewhat obscure proof that 3-dimensional positive definite quadratic forms are determined by their representation numbers, and discuss some related open problems. Schiemann's aforementioned proof was originally given in 1994 and it leaves a lot of work for the reader. We hope to iron out these details and to present the different equivalent perspectives of the problem in a comprehensible way.

As we shall see, this subject is at the intersection of analysis, geometry, number theory and computer science. I personally find a lot of enjoyment in using multiple areas of mathematics to solve problems; it makes the results more rewarding and somehow more profound. Most of these connections will be discussed in section 0.2. There, we most importantly explain the equivalence of loosely speaking determining the spectral geometry of flat tori and deciding whether the image of positive definite quadratic forms determines them. All of sections 0.1, 0.2 and 0.3 might be best understood after having read the rest of the thesis, but hopefully they serve as a sensible overview of what is to come.

### 0.1 Three Different Perspectives

The three perspectives, numbered as 1,2 and 3 in this section, all give their unique contributions to this thesis and most importantly, they are equivalent as will be explained in the next section. The first is the analytical, Riemannian perspective whose implications are explored in section 2.3. The second is that of the number theory of quadratic forms which is vital for proving Schiemann's theorem, which is state at the end of this section. The third is that of lattice geometry which gives us a great intuition for how to solve simpler problems and even more difficult ones, and is sometimes more convenient to work with. All of this is done in order to explain the different ways you can view the problem that Schiemann solved.

For now, we define concepts and give statements that we will return to and explain in later chapters. We may think of this section as a summary of all the basic ideas in this paper. Let's first look at the two perspectives that lie at the heart of this thesis. They can be expressed through the following two questions,

- 1. Do the spectra of the Laplace operator determine the geometry of flat tori?
- 2. Are quadratic positive definite forms determined by the multiplicities of their integral values?

To understand these problems, we need to know what a *lattice* is. An  $n$ -dimensional (full-rank) lattice  $\Gamma$  is a set  $\Gamma := AZ^n$  for some invertible matrix  $A \in GL_n(\mathbb{R})$ . We say that  $A$  is a basis matrix of  $\Gamma$ . Further, a *flat torus* is a quotient space  $\mathbb{T}_\Gamma := \mathbb{R}^n/\Gamma$  for some  $n$ -dimensional lattice  $\Gamma$ . We recall from topology that a 2-dimensional torus is embedded in 3 dimensions by taking a parallelogram and identifying its edges such that we first get a cylinder, and then glue the two remaining edges together. In this case, it is the parallelogram that is the corresponding flat torus (still with identified edges). Similarly in higher dimensions, we view a flat torus as a parallelepiped with identified facets.

Let  $\Gamma_1 = A_1\mathbb{Z}^n, \Gamma_2 = A_2\mathbb{Z}^n$  be two lattices. We have  $\Gamma_1 = \Gamma_2$  if and only if  $A_2 = A_1B$  for some  $B \in GL_n(\mathbb{Z})$  as is seen in chapter 1, which we devote to lattice theory. This right multiplication by unimodular matrices gives an equivalence relation on bases. Further, we view two flat tori  $\mathbb{T}_{\Gamma_1}, \mathbb{T}_{\Gamma_2}$  as having the same shape, or as *congruent*, if  $\Gamma_2 = C\Gamma_1$  for some orthogonal matrix  $C$ . It is natural to let left multiplication by elements of  $O_n(\mathbb{R})$  define an equivalence relation on basis matrices of lattices and by doing so we get that

$$GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n(\mathbb{R}) \quad (\star)$$

can be identified as the set of all lattices of different shape, or similarly all flat tori of different shape. We also have a notion of equivalence of quadratic forms, which are functions  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  on the form  $q(x) = x^T Q x$  for some matrix  $Q \in \mathbb{R}^{n \times n}$ . We refer to chapter 4 for a more detailed description. We say that  $q_1, q_2$  are equivalent if for some unimodular matrix  $B \in GL_n(\mathbb{Z})$ , we have  $q_2(x) = q_1(Bx)$  for all  $x \in \mathbb{Z}^n$ . On this note, we define  $\delta_n^+$  to be the set of symmetric quadratic positive definite forms in dimension  $n$ , viewing it both as the set of forms and corresponding matrices. We can now identify the set of all different symmetric positive definite quadratic forms as

$$\delta_n^+ / GL_n(\mathbb{Z}). \quad (\#)$$

We have now classified all *distinct* lattices, flat tori and all *distinct* positive definite forms. Importantly, the Cholesky decomposition of positive definite matrices gives us a bijection between the sets  $(\star)$  and  $(\#)$ , which is explained in the next section. The decomposition says that to each symmetric positive definite matrix  $Q$ , there is an invertible matrix  $A$  with  $A^T A = Q$ . Further, for any invertible matrix  $A$ ,  $A^T A$  is a symmetric positive definite matrix. Let  $q$  be a positive definite quadratic form with  $Q$  as its matrix. The values  $q(x)$  can all be written on the form  $x^T Q x = (Ax)^T (Ax) = \|Ax\|^2$  for some  $A$ . Throughout this report we will denote the Euclidean norm with  $\|\cdot\|$ . Because of this, we can observe the similarities in the following definitions. For a lattice  $\Gamma = A\mathbb{Z}^n$ , we define its *length spectrum* to be

$$\mathcal{L}_\Gamma := \{(\lambda, m) : 0 \neq m = \#\{\gamma \in \Gamma : \lambda = \|\gamma\|\}\}.$$

For a positive definite form  $q$  we define its *representation numbers* to be the values of  $t \in \mathbb{R}_0^+$ , where  $\mathbb{R}_0^+$  is the set of all non-negative real numbers, given by the function

$$\mathcal{R}(q, t) := \#\{x \in \mathbb{Z}^n : q(x) = t\}.$$

We say that  $\#\emptyset = 0$  as a convention. To see the relevance of the length spectrum, we first look at the spectra of the Laplace operator on flat tori. The problem is to find functions on, say  $\mathbb{T}_\Gamma$ , satisfying certain conditions and values  $\lambda$  such that

$$-\Delta f = \lambda f.$$

We then define  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$  to be the set of pairs of  $(\lambda, m)$  with  $0 \neq m = \dim E_\lambda$ , where  $E_\lambda$  is the eigenspace of  $\lambda$ . It turns out that two flat tori are *isospectral*, meaning that their Laplace spectra are equal, if and only if their length spectra are equal. This is explained in detail in chapter 2, where we also see that the dimension of the flat torus is determined by its Laplace spectrum. The core of this thesis is the question of spectral determination. To answer whether the Laplace operator determines the geometry of flat tori, we really do mean whether or not flat tori are *spectrally determined*. A flat torus is spectrally determined if any flat torus that is isospectral to it, must also be isometric to it in the Riemannian sense (and by theorem 2.0.3 they are isometric if and only if their corresponding lattices are congruent).

Let us finally give the third perspective. In light of the properties of the length spectrum and since it is the set of lengths with multiplicities of points in the underlying lattice, we note that we can rephrase the question of spectral determination of flat tori as the following question for their underlying lattices:

- 3. Do the lengths of points of a lattice with multiplicity determine the lattice itself up to congruency?

We now state the result that answers the three questions that we have posed. This result is not widely known as ‘‘Schiemann’s theorem’’, but we thought it was a fitting name. The proof is divided into multiple parts which we explain in section 0.3.

**Theorem** (Schiemann’s Theorem). *The answer to the questions 1,2 and 3 is yes if and only if we are in dimension 3 or less.*

## 0.2 The Equivalence

To see that the three questions that were posed in the prior section are indeed equivalent, we first rephrase questions 1 and 2 as follows,

- 1. Are  $n$ -dimensional flat tori spectrally determined?
- 2. Are elements of  $\delta_n^+$  determined by their representation numbers?

In the previous section we explained why questions 1 and 3 were equivalent, so we are left to prove that 1 and 2 are equivalent. To do this we start by constructing a formal bijection from the set of distinct flat tori, identified with the basis matrices of their underlying lattices, to quadratic positive definite forms by,

$$\rho : GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n(\mathbb{R}) \rightarrow \delta_n^+ / GL_n(\mathbb{Z}),$$

$$A \mapsto A^T A.$$

This function is well-defined since if we take any basis matrix  $A$  and  $B \in GL_n(\mathbb{Z}), C \in O_n(\mathbb{R})$ , then  $CAB \mapsto B^T A^T C^T CAB = B^T A^T AB$  which is an equivalent form to  $A^T A$  in  $\delta_n^+$ . To see injectivity, if  $A^T A = A'^T A'$ , then  $I = (A'A^{-1})^T A'A^{-1}$ . In other words  $A' = CA$  for some  $C \in O_n(\mathbb{R})$  which implies  $A, A'$  are equivalent in the quotient space. To see surjectivity, we only need to refer to Cholesky decomposition.

By noting that  $\mathcal{R}(\rho(L), t)$  is equal to the number of vectors  $Lx$  such that  $\|Lx\| = t$  for  $x \in \mathbb{Z}^n$ , we observe that the length spectra of  $\Gamma = A\mathbb{Z}^n, \Gamma' = A'\mathbb{Z}^n$  are equal if and only if the representation numbers of  $\rho(A), \rho(A')$  are equal. The lattices  $\Gamma, \Gamma'$  are congruent if and only if  $\rho(A), \rho(A')$  are equivalent, since congruency here means precisely that  $A, A'$  are in the same equivalence class in the quotient space. So if we find that the two flat tori  $\mathbb{T}_\Gamma, \mathbb{T}_{\Gamma'}$  are isospectral, but non-congruent, then we have found two forms in  $\delta_n(\mathbb{R})$ , namely  $A^T A, A'^T A'$  that have equal representations numbers, but are different forms and vice versa. Due to this equivalence, we only have to write one of the perspectives when formulating Schiemann's theorem.

## 0.3 Schiemann's Theorem I & IV

We may separate Schiemann's theorem into four distinct parts as follows: Part I is for  $n = 1$ , part II is for  $n = 2$ , part III is for  $n = 3$  and part IV is for  $n \geq 4$ . Part II will be discussed in section 3.1 and part III will be discussed in chapter 5. The proof for the first part is trivial.

**Theorem 0.3.1** (Schiemann's Theorem I). *1-dimensional lattices are determined by their lengths with multiplicity.*

*Proof.* All 1-dimensional lattices can be described by  $\lambda\mathbb{Z} = \{\lambda z : z \in \mathbb{Z}\}$  for some  $\lambda > 0$ . If the lengths of the lattices  $\lambda_1\mathbb{Z}$  and  $\lambda_2\mathbb{Z}$  are equal, then they must have their shortest vector in common. This implies directly that  $\lambda_1 = \lambda_2$  since we choose  $\lambda_1, \lambda_2 > 0$ , showing that the lattices must be the same and are therefore congruent.  $\odot$

The symbol  $\odot$ , resembling a torus (that is not flat!), will be the Q.E.D symbol throughout this paper. Schiemann's theorem part II can also be shown in an elementary way which we discuss in chapter 3. Part III is however not so easy to prove; it will be the subject of chapter 5 where we must apply computer algorithms.

**Theorem 0.3.2** (Schiemann's Theorem IV). *As long as  $n \geq 4$ ,  $n$ -dimensional positive definite forms are not determined by their values and multiplicities over  $\mathbb{Z}^n$ .*

Interestingly, both parts III and IV of Schiemann's theorem were proven by Alexander Schiemann between the years of 1990-1994 with the perspective of quadratic forms [1] [2]. Unfortunately however, he stopped working with mathematics shortly after. Historically, the first pair of isospectral non-isometric flat tori was 16-dimensional and was found by Milnor by building on findings of Witt [13]. Over time, a 12-dimensional example was found by Kneser in 1967 and an 8-dimensional one was found by Kitaoka in 1977 [14] [15]. The 12-dimensional example will later serve as motivation for why the conjecture in section 3.3 shouldn't hold in greater generality.

To show part IV of Schiemann's theorem, it is enough to find a pair of 4-dimensional isospectral non-isometric flat tori, since we can then refer to Schiemann's lemma in section 2.4, and that's what Schiemann did. The problem of finding pairs of 4-dimensional isospectral non-isometric pairs of flat tori was later expanded on by Conway and Sloane who in 1992 found a big family of such tori, that includes Schiemann's example [4]. Quite recently in 2009 Cervino and Hein showed that there are in fact an infinite number of distinct pairs of such tori in 4 dimensions by building upon Conway and Sloane's findings [9].

## 0.4 Reader's Guide

This thesis presents all the most important results regarding Schiemann's theorem and questions surrounding it. A reader who is experienced in the field might want to skip the three first sections entirely, even though there might be some noteworthy statements in sections 1.4 and 2.4. Section 1.1 about linear algebra could admittedly be omitted, but it serves as a familiar introduction. For the remainder of section 1, we go over the basics of lattice theory and some more involved statements in sections 1.3 and 1.4, among which we discuss Minkowski reduction which we will return to in chapter 4. The original analytic perspective comes from Riemannian geometry and will be mentioned in chapter 2. In terms of analysis, we will put our focus on methods from analysis that work wonders in section 2.3. We continue in chapter 3 by showing and stating important results that are related to Schiemann's theorem. For the reader that only wishes to read the proof for part III of Schiemann's theorem, it is enough to read chapters 4 and 5. Little to no prior information is needed. The final chapter 6 gives some ideas for how to solve unsolved problems.

# Chapter 1

## Lattice Theory

As we observed in the previous chapter, Schiemann's theorem is really the answer to the question,

- Do the lengths of points of a lattice with multiplicity determine the lattice itself up to congruency?

For this reason alone, it makes sense to develop some theory on lattices. The purpose of this chapter is to provide some fundamental concepts and simple proofs that will lay the groundwork for future sections. For example, the dual lattice will be of importance in chapter 2 and its connection to the Laplace spectrum of flat tori will be made clear. To begin with however, we state some results about linear algebra that we will apply. Later we will look at the basics of lattice theory and Minkowski reduction. Finally, we give a proof for how congruence and products of lattices relate in section 1.4. We recall the definition of a lattice,

**Definition 1.0.1** (Lattice). *An  $n$ -dimensional (full-rank) lattice  $\Gamma$  is the set  $\Gamma := AZ^n$  for some invertible matrix  $A \in \mathbb{R}^{n \times n}$ . The matrix  $A$  is called a basis matrix of  $\Gamma$ .*

We observe that a lattice is an additive group with the identity  $0 \in AZ^n$ . If we don't say anything else, a lattice will always refer to a full-rank lattice. However, we will later look at lattices that are not of full-rank.

### 1.1 Recalling Linear Algebra

Most of the statements we present in this section are of course well-known, and they are stated for the sake of reference. We will in particular discuss the following three matrix groups and formally introduce positive definite quadratic forms.

**Definition 1.1.1** (General Linear, Unimodular and Orthonormal Matrix Groups). *We define the general linear group, the unimodular group, and the orthonormal matrix group, respectively, as the sets,*

$$\begin{aligned} GL_n(\mathbb{R}) &:= \{A \in \mathbb{R}^{n \times n} : \det(A) \neq 0\}, \\ GL_n(\mathbb{Z}) &:= \{B \in \mathbb{Z}^{n \times n} : \det(B) \neq 0 \ \& \ B^{-1} \in \mathbb{Z}^{n \times n}\}, \\ O_n(\mathbb{R}) &:= \{C \in \mathbb{R}^{n \times n} : C^T C = I\}. \end{aligned}$$

The usual term for  $O_n(\mathbb{R})$  is the orthogonal group, but I will refer to it as the *orthonormal* group since it makes it more clear to us what it means. All three are groups with respect to matrix multiplication, where the identity element  $I$  is in all three sets.

**Lemma 1.1.2.** *The set  $O_n(\mathbb{R})$  consists of matrices whose column vectors form an orthonormal basis of  $\mathbb{R}^n$ . Further, let  $A$  be linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Then  $A \in O_n(\mathbb{R})$  if and only if  $A$  takes an orthonormal basis to another.*

The proofs for this lemma and the next proposition are given in appendix A. One should think of the orthonormal group as the set of matrices that are either rotations, reflections or compositions of these. We can make use of this group to rewrite invertible matrices as follows.

**Proposition 1.1.3.** *To any  $A \in GL_n(\mathbb{R})$ , there is a  $C \in O_n(\mathbb{R})$  such that  $CA$  is an upper triangular matrix with positive diagonal elements.*

The following is a well-known characterizations of the matrix groups  $GL_n(\mathbb{Z})$  and  $O_n(\mathbb{R})$ . They will be very useful going forward. From now on,  $\cdot$  refers to the standard inner product on  $\mathbb{R}^n$ .

**Proposition 1.1.4.**

- 1) Let  $B$  be any real  $n \times n$  matrix. The following are equivalent:
  - i)  $B \in GL_n(\mathbb{Z})$
  - ii)  $\det(B) = \pm 1$  and  $B \in \mathbb{Z}^{n \times n}$ ,
- 2) Let  $C$  be any real  $n \times n$  matrix. The following are equivalent:
  - i)  $C \in O_n(\mathbb{R})$
  - ii) Each vector of  $C$  is of length 1, and its column vectors are pairwise orthogonal,
  - iii)  $C$  preserves distance,
  - iv)  $Cx \cdot Cy = x \cdot y$  for all  $x, y \in \mathbb{R}^n$ ,
  - v) Each column vector of  $C$  is of length 1 and  $\det(C) = \pm 1$ .

The content of the above proposition should be familiar, except for maybe the last part. It is intuitively clear however that 2v) holds if and only if 2ii) does; if we inscribe  $n$  unit vectors into the unit sphere  $S_{n-1}$ , then the parallelepiped spanned by those vectors can only have volume 1 if it is an  $n$ -cube up to rotation. We finally move on to positive definite quadratic forms and Cholesky decomposition.

**Definition 1.1.5** (Positive Definite Quadratic Forms). *A positive definite quadratic form is a multivariate polynomial  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  that is defined by*

$$q(x) := x^T Q x,$$

where  $Q \in \mathbb{R}^{n \times n}$  is a positive definite matrix. In other words  $x^T Q x > 0$  for all non-zero  $x \in \mathbb{R}^n$ .

We might as well assume that  $Q$  should be symmetric. Since we have  $x^T Q x = (x^T Q x)^T = x^T Q^T x$ , the matrices  $Q$  and  $(Q^T + Q)/2$  give the same values as quadratic forms and the latter is symmetric.

**Theorem 1.1.6** (Cholesky Decomposition). *To each symmetric positive definite matrix  $Q$ , there is an invertible triangular matrix  $A$  with*

$$A^T A = Q$$

Further, for any invertible matrix  $A$ ,  $A^T A$  is a symmetric positive definite matrix.

*Proof.* It is clear that if  $n = 1$ , then  $Q = q_{11} > 0$  and we can let  $A = \pm\sqrt{q_{11}}$ . Now assume that a decomposition exists for all symmetric positive definite matrices up until dimension  $n - 1$ . Consider a symmetric positive definite matrix  $Q \in \mathbb{R}^{n \times n}$ . Let  $Q_0$  be the  $n - 1 \times n - 1$  upper left submatrix of  $Q$ . By assumption, there is an invertible triangular matrix  $A_0$  such that  $A_0^T A_0 = Q_0$ . For some  $b \in \mathbb{R}^{n-1}$  we define,

$$A := \begin{bmatrix} A_0 & b \\ 0^T & b_n \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} Q_0 & A_0^T b \\ b^T A_0 & b^T b + b_n^2 \end{bmatrix}.$$

We only need to show that  $b$  and  $b_n$  can be chosen such that  $Q = A^T A$ . Now  $A_0^T$  is invertible, meaning that we can always find an appropriate choice of  $b$ . We are left to show that  $q_{nn} > b^T b$  so that we can let  $b_n = \pm\sqrt{q_{nn} - b^T b}$ . Assume  $q_{nn} \leq b^T b$ . Then consider for  $0 \neq x \in \mathbb{R}^n$  the following,

$$0 < x^T Q x = x^T \begin{bmatrix} Q_0 & A_0^T b \\ b^T A_0 & q_{nn} \end{bmatrix} x \leq x^T \begin{bmatrix} A_0^T A_0 & A_0^T b \\ b^T A_0 & b^T b \end{bmatrix} x = x^T \begin{bmatrix} A_0^T & 0 \\ b^T & 0 \end{bmatrix} \begin{bmatrix} A_0 & b \\ 0 & 0 \end{bmatrix} x.$$

The right hand side clearly is zero for a non-zero choice of  $x$ , which gives a contradiction. Therefore  $q_{nn} > b^T b$  and we are done. For the last part, note that  $(A^T A)^T = A^T A$  and  $x^T A^T A x = (Ax)^T Ax = \|Ax\|^2 > 0$  for  $x \neq 0$  since  $A$  is invertible. ◻

For a matrix  $Q$ , we usually denote its entries by  $q_{ij}$  where  $i$  denotes its row and  $j$  its column, meaning that we write  $Q = (q_{ij})$ . Not until chapter 4 will we give an in-depth description of symmetric positive definite forms. For now it is enough with the following lemma.

**Lemma 1.1.7.** *A quadratic form  $q(x) = x^T Q x$  is determined by the values of  $e_i + e_j$  for  $1 \leq i, j \leq n$ .*

*Proof.* Let  $Q = (q_{ij})$ . We have  $q(e_i + e_j) = e_i^T Q e_i + 2e_i^T Q e_j + e_j^T Q e_j = q_{ii} + 2q_{ij} + q_{jj}$ . In particular if  $i = j$ , then  $q(2e_i) = 4q_{ii}$  which determines the diagonal elements. Then the above formula determines  $q_{ij}$  for  $i \neq j$ .  $\odot$

## 1.2 Bases, Products & Duals

With the recap of linear algebra out of the way, we show a number of basic results for lattices that we need. We will also introduce concepts such as the product of lattices and the dual lattice. Let us begin with an intuitively clear result about lattices.

**Proposition 1.2.1.** *An  $n$ -dimensional lattice is a closed, discrete set in  $\mathbb{R}^n$ . Moreover, the set  $\{\|Ax\| : x \in \mathbb{Z}^n\}$  is closed and discrete.*

*Proof.* Let  $X \subseteq \mathbb{R}^k$  be a set such that any sequence  $x_i \in X$  that converges in  $\mathbb{R}^k$  becomes stationary. If  $X$  were not discrete, we would directly find a contradiction. It is also clear that  $X$  is closed since it contains all its limit points. With this criterion, we continue as follows. Consider an arbitrary sequence  $Ax_i$  for  $x_i \in \mathbb{Z}^n$  which converges to a limit point  $z \in \mathbb{R}^n$ . By the invertibility of  $A$  and the continuity of its inverse,  $x_i \rightarrow A^{-1}z$ . Since we around each point of the lattice  $\mathbb{Z}^n$  can place a ball of radius say  $1/2$  that intersects no other point, the sequence  $x_i$  must become stationary. It follows that  $Ax_i$  becomes stationary. For the second part, we instead consider any sequence  $\|Ay_i\|$  for  $y_i \in \mathbb{Z}^n$  which converges to some  $z \in \mathbb{R}$ . Assume by contradiction that it does not become stationary. Then we can find a subsequence  $\|Ay_{i_j}\|$  with distinct lengths for each  $j$ . We can find yet another subsequence such that  $Ay_{i_{j_m}}$  converges in  $\mathbb{R}^n$ . By what we have done previously, it follows that this sequence becomes stationary, which contradicts that each length  $\|Ay_{i_j}\|$  should be unique.  $\odot$

Let  $\Gamma = AZ^n$  be some lattice with  $A = [a_j]$ , where  $[a_j]$  is the column vector notation for matrices. It is clear that for any  $\sigma \in S_n$ , we have  $[a_{\sigma(j)}]Z^n = [a_j]Z^n$ , basically since addition in  $\mathbb{R}^n$  is commutative. Considering the standard basis  $e_i$  for  $\mathbb{R}^n$ , note that the matrix multiplication  $[a_j][e_{\sigma(j)}]$  is equal to  $[a_{\sigma(j)}]$  where  $[e_{\sigma(j)}] \in GL_n(\mathbb{Z})$ . This observation can be generalized as follows:

**Proposition 1.2.2.** *For two lattices  $\Gamma_1 = A_1Z^n, \Gamma_2 = A_2Z^n$  we have  $\Gamma_1 = \Gamma_2$  if and only if  $A_2 = A_1B$  for some  $B \in GL_n(\mathbb{Z})$ .*

*Proof.*

$\Rightarrow$ ) For each  $\alpha \in \mathbb{Z}^n$  there is a  $\beta(\alpha) \in \mathbb{Z}^n$  with  $A_1\alpha = A_2\beta(\alpha)$  which implies  $\beta(\alpha) = A_2^{-1}A_1\alpha$ . Now  $A_2^{-1}A_1$  is a bijective linear transformation  $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$ , therefore  $\beta(e_i) \in \mathbb{Z}^n$ . It follows that each vector of  $B_1 := A_2^{-1}A_1$  is in  $\mathbb{Z}^n$ . By swapping  $A_1, A_2$  in the same argument, we also get that each vector of  $B_2 := A_1^{-1}A_2$  is in  $\mathbb{Z}^n$ . Now  $B_2B_1 = I$  and  $\det(B_1), \det(B_2) \in \mathbb{Z}$ , which implies  $\det(B_1), \det(B_2) \in \{-1, 1\}$ .

$\Leftarrow$ ) To show  $\Gamma_1 = \Gamma_2$  we only need to see that  $\mathbb{Z}^n = BZ^n$  for a matrix  $B \in GL_n(\mathbb{Z})$ .  $\odot$

**Definition 1.2.3** (Dual lattice). *For a lattice  $\Gamma$  we define its dual to be,*

$$\Gamma^* := \{\gamma^* \in \mathbb{R}^n : \gamma^* \cdot \gamma \in \mathbb{Z}, \forall \gamma \in \Gamma\}.$$

It makes sense to call this set the dual because of the lemma A.0.3, that is a consequence of the Riesz representation theorem. Now that we have motivated calling it a “dual” we must motivate that it is a lattice.

**Proposition 1.2.4.** *If  $\Gamma = AZ^n$ , then  $\Gamma^* = A^{-T}Z^n$ . It follows that  $\Gamma^*$  is a lattice.*

*Proof.* We have by definition  $\Gamma^* = \{\gamma^* \in \mathbb{R}^n : \gamma^* \cdot A\alpha \in \mathbb{Z}, \forall \alpha \in \mathbb{Z}^n\}$ . First note that  $\gamma^* \cdot A\alpha \in \mathbb{Z}$  if and only if  $(A^T\gamma^*)^T\alpha \in \mathbb{Z}$ . Choosing  $\alpha = e_i$ , we see that  $(A^T\gamma^*)_i \in \mathbb{Z}$  for all  $i$  which shows  $A^T\gamma^* \in \mathbb{Z}^n$ . Hence  $\gamma^* \cdot A\alpha$  is equivalent to  $A^T\gamma^*$ . By the invertibility of  $A^T$ , the set of  $\gamma^* \in \mathbb{R}^n$  with  $A^T\gamma^* \in \mathbb{Z}^n$  is precisely  $A^{-T}Z^n$ .  $\odot$



We now turn our attention to products of lattices, which we will talk more in-depth about in sections 1.4 and 3.3. Historically its importance has been evident in the search to find isospectral non-isometric flat tori in different dimensions.

**Definition 1.2.5** (Product of lattices). *Let  $\Gamma, \Gamma'$  be two lattices. We define their their product in the natural way to be the following set,*

$$\Gamma \times \Gamma' := \{(\gamma, \gamma') : \gamma \in \Gamma \ \& \ \gamma' \in \Gamma'\}.$$

Observe that the product is indeed a lattice, which we now shall motivate. If  $\Gamma = A\mathbb{Z}^n, \Gamma' = A'\mathbb{Z}^m$  are two lattices, then by definition we get the following equality,

$$\Gamma \times \Gamma' = \{(A\alpha, A'\beta) : (\alpha, \beta) \in \mathbb{Z}^n \times \mathbb{Z}^m\} = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \mathbb{Z}^{n+m}.$$

So we have created a new lattice. We write  $\Gamma^n$  to denote a product consisting of precisely  $n$  lattices  $\Gamma$ . Using proposition 1.9 we also find the following base for the dual of a product,

$$(\Gamma \times \Gamma')^* = \begin{bmatrix} A^{-T} & 0 \\ 0 & A'^{-T} \end{bmatrix} \mathbb{Z}^{n+m}.$$

In other words, we have shown that  $(\Gamma \times \Gamma')^* = \Gamma^* \times \Gamma'^*$ . As preparation for section 1.4, we will recall the concept of *congruent* lattices. To motivate the use of the word *congruence*, we might think of congruent triangles. Two congruent triangles have equal edge lengths and equal angles in such a way that they only differ by a rotation or reflection. Formally we define,

**Definition 1.2.6** (Congruent Lattices). *Two lattices  $\Gamma_1, \Gamma_2$  are congruent if  $\Gamma_2 = C\Gamma_1$  for an orthonormal transformation  $C \in O_n(\mathbb{R})$  for some  $n \in \mathbb{N}$ . We write  $\Gamma_1 \stackrel{C}{\sim} \Gamma_2$  to denote congruency.*

It follows directly that even though we didn't require  $\Gamma_1, \Gamma_2$  to be of the same dimension, congruency implies that they are. We end by formalizing an observation that we will return to later.

**Lemma 1.2.7.** *Let  $\Gamma_1 = A_1\mathbb{Z}^n, \Gamma_2 = A_2\mathbb{Z}^n$  be two lattices. The following are equivalent,*

- i)  $\Gamma_1 \stackrel{C}{\sim} \Gamma_2$ ,
- ii)  $A_2 = CA_1B$  for some  $C \in O_n(\mathbb{R}), B \in GL_n(\mathbb{Z})$ ,
- iii)  $B = A_2^{-1}CA_1$  for some  $B \in GL_n(\mathbb{Z}), C \in O_n(\mathbb{R})$ .

*Proof.*

$i) \Leftrightarrow ii)$  : That  $\Gamma_1, \Gamma_2$  are congruent means precisely that  $\Gamma_2 = C\Gamma_1$  for some  $C \in O_n(\mathbb{R})$ . It follows that  $CA_1$  is a basis matrix for  $\Gamma_2$  and by proposition 1.2.2, this is true if and only if  $A_2 = CA_1B$  for some  $B \in GL_n(\mathbb{Z})$ .

$ii) \Leftrightarrow iii)$  : This is direct since  $iii)$  is simply a rewriting of  $ii)$ .  $\square$

**Definition 1.2.8** (Fundamental Domain,  $R_A$ ). *A fundamental domain for an  $n$ -dimensional lattice  $\Gamma$  is the parallelootope spanned by one of its basis matrices  $A$ . More precisely,*

$$R_A := \{Ax : x \in [0, 1]^n\}.$$

The fundamental domain is not unique; it is different for each different choice of basis matrix. But no matter which basis we choose, we clearly have if  $R_A + \gamma$  denotes a Minkowski sum,

$$\bigcup_{\gamma \in \Gamma} R_A + \gamma = \mathbb{R}^n.$$

### 1.3 Minkowski Reduction

A very important notion in Lattice theory is the Minkowski reduction of bases, which equivalently exists for quadratic forms. It is the task of finding an "optimal" basis. We will expand on the theory of Minkowski reduction in chapter 4 to help us prove the third part of Schiemann's theorem.

**Definition 1.3.1** (Minkowski Reduced Basis of a Lattice). Consider an  $n$ -dimensional lattice  $\Gamma$ . A basis matrix  $A = [a_j]$  of  $\Gamma$  is Minkowski reduced if for each  $j$ ,  $a_j$  is a shortest choice of vector in  $\Gamma$  such that  $a_1, \dots, a_j$  is part of some basis  $a_1, \dots, a_i, f_{i+1}, \dots, f_n$  of  $\Gamma$ .

Sometimes the condition  $0 \leq a_j \cdot a_{j-1}$  for  $1 < j \leq n$  is also given for Minkowski reduced bases, but we can always recover this by simply changing the signs of the vectors. The Minkowski reduction is either way not unique. If  $a_1$  is a possible choice of the shortest vector in  $\Gamma$ , then clearly  $-a_1$  is as well. Saying that the vectors  $a_1, \dots, a_i$  is part of some basis is also referred to as them being *extensible* to a basis. Before we define Minkowski reduction on positive definite forms, we give a motivation using the following lemma.

**Lemma 1.3.2.** Let  $f_1, \dots, f_n$  be a basis for  $\Gamma$  and let  $f = \sum_1^n \lambda_j f_j \in \Gamma$  with  $\lambda_j \in \mathbb{Z}$ . If  $1 \leq m < n$ , then the following are equivalent:

- i) The vectors  $f_1, \dots, f_{m-1}, f$  are extensible to a basis for  $\Gamma$ ,
- ii)  $u_1 f_1 + \dots + u_{m-1} f_{m-1} + u_m f \in \Gamma$  implies that  $u_i$  are integers,
- iii)  $\text{GCD}(\lambda_m, \dots, \lambda_n) = 1$ .

*Proof.* See [6, p. 14]. ⊙

**Lemma 1.3.3.** Let  $Q$  be a symmetric positive definite quadratic form and  $A$  be its Cholesky decomposition such that  $Q = A^T A$ . Then  $A$  is Minkowski reduced as a basis matrix of  $\Gamma = A\mathbb{Z}^n$  if and only if the following holds for  $Q = (q_{ij})$ : for all  $k = 1, \dots, n$  and for all  $x \in \mathbb{Z}^n$  with  $\text{GCD}(x_k, \dots, x_n) = 1$ , we have  $q(x) \geq q_{kk}$ .

*Proof.*

$\Rightarrow$ ) Assume by contradiction that the statement is false; for some  $x \in \mathbb{Z}^n$  and  $k$  with  $\text{GCD}(x_k, \dots, x_n) = 1$  we have  $q(x) < q_{kk}$ . Since we can by lemma 1.3.2 extend  $a_1, \dots, a_{k-1}, Ax$  to a basis for  $\Gamma$  and  $q(x) = \|Ax\|^2 < \|Ae_k\|^2 = \|a_k\|^2 = q_{kk}$ ,  $A$  could not have been Minkowski reduced to begin with.

$\Leftarrow$ ) We have  $\|Ax\| \geq \|Ae_k\| = \|a_k\|$  for each  $x \in \mathbb{Z}^n$  with  $\text{GCD}(x_k, \dots, x_n) = 1$ , therefore by lemma 1.3.2, if the set of vectors  $a_1, \dots, a_{k-1}, Au$ , for some  $u \in \mathbb{Z}^n$  could be extended to a basis of  $\Gamma$ , then it would follow by assumption that  $\|Au\| \geq \|a_k\|$ . Therefore, each  $a_k$  is a shortest choice which we wanted. ⊙

**Definition 1.3.4.** A positive definite quadratic form  $q$  is Minkowski reduced if for all  $k = 1, \dots, n$  and for all  $x \in \mathbb{Z}^n$  with  $\text{GCD}(x_k, \dots, x_n) = 1$ , we have  $q(x) \geq q_{kk}$ .

**Theorem 1.3.5** (Existence of Reduction). To each positive definite quadratic form there is a finite non-zero number of equivalent Minkowski reduced forms.

*Proof.* See [6, p. 27-28]. ⊙

For the reader who is interested in looking up the proof, we state and prove proposition A.0.5 that Cassels partly leaves up to the reader. As a direct observation of lemma 1.3.2 when considering the standard basis  $f_i = e_i$  of  $\Gamma = \mathbb{Z}^n$ , we find that any  $x$  with  $\text{GCD}(x_i) = 1$  can be extended to a basis of  $\mathbb{Z}^n$ :

**Corollary 1.3.6** (Bezout's Theorem, Alternate Version). Let  $x \in \mathbb{Z}^n$  be a vector such that  $\text{GCD}(x_1, \dots, x_n) = 1$ , then there exists a matrix  $B \in GL_n(\mathbb{Z})$  with  $x$  as its first vector.

The connection to Bezout's theorem is clear; since  $\det(B) = \pm 1$ , when we expand the determinant with respect to the first vector, we get a  $y \in \mathbb{Z}^n$  such that  $y^T x = 1$ . In any case, this guarantees that the first column vector of a Minkowski reduced basis matrix is a shortest non-zero vector of the lattice.

We might ask ourselves if there is a better and more intuitive reduction. For example, we would optimally like to find that a basis matrix  $A = [a_j]$  for any lattice  $\Gamma$  can be defined such that each

$$a_j \in \Gamma \setminus \text{Span}\{0, a_1, \dots, a_{j-1}\} \quad (\star)$$

is any shortest choice of vector. However this does not hold in general, which is *extremely* unfortunate. It does however hold as long as we are in 3 or less dimensions as we see in theorem 1.3.7. We now give a concrete example of what happens in higher dimensions. Consider the basis matrix

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

of the lattice  $\Gamma = A_4\mathbb{Z}^4$ , with the column vector notation  $A_4 = [a_j]$ . We see that  $e_4 = -a_1 - a_2 - a_3 + 2a_4$  is of the same length as  $a_4$  and is linearly independent of  $a_1, a_2, a_3$ . However the system  $a_1, a_2, a_3, e_4$  is not a basis for  $\Gamma$ . In general for  $n \geq 5$ , let  $A_n = [e_1, \dots, e_{n-1}, \frac{1}{2}\mathbf{1}]$ , where  $\mathbf{1} = e_1 + \dots + e_n$ . Then  $e_n \in A_n\mathbb{Z}^n$  is shorter than  $\frac{1}{2}\mathbf{1}$ , but  $e_1, \dots, e_n$  is not a basis for  $A_n\mathbb{Z}^n$ .

**Theorem 1.3.7** (Intuitive Reduction). *As long as  $n \leq 3$ , we can find a basis matrix  $A = [a_j]$  for any  $n$ -dimensional lattice  $\Gamma$  such that  $a_1$  is a shortest non-zero vector of  $\Gamma$  and each  $a_i$  is ANY shortest choice such that  $a_1, \dots, a_i$  are linearly independent. If  $n = 4$ , then we have the equivalent if we replace ANY with SOME.*

*Proof.* See [16, p. 278]. ◻

It follows that any two different Minkowski reduced basis matrices  $[a_j], [a'_j]$  for a lattice of dimension 3 or lower must have  $\|a_j\| = \|a'_j\|$  for  $j = 1, 2, 3$ . In terms of the Minkowski reduction of quadratic forms, this says that if  $q \in \delta_n^+$  for  $n \leq 3$  is Minkowski reduced, then  $q \circ B$  is Minkowski reduced if and only if it has the same diagonal elements as  $q$ , where  $B \in GL_n(\mathbb{Z})$ .

We give a simple proof for this intuitive reduction in two dimension. Let  $\Gamma = AZ^2$  be a 2-dimensional lattice and let  $a_1, a_2$  be shortest choices of vectors as in  $(\star)$ . By assuming that there is a point  $\gamma$  in  $\Gamma \setminus \text{Span}_{\mathbb{Z}}\{a_1, a_2\}$  we will come to a contradiction. Since  $\text{Span}_{\mathbb{Z}}\{a_1, a_2\}$  is a lattice, the discussion at the end of section 1.2 says that for  $R_{[a_1, a_2]} = \{\lambda_1 a_1 + \lambda_2 a_2 : 0 \leq \lambda_1, \lambda_2 \leq 1\}$ ,

$$\bigcup_{\gamma \in \Gamma} R_{[a_1, a_2]} + \gamma = \mathbb{R}^2.$$

In particular, since  $a_1, a_2 \in \Gamma$  we can choose our  $\gamma$  to be inside the set  $R_{[a_1, a_2]}$ . By the triangle inequality,  $\|a_1 + a_2\| \leq 2\|a_2\|$  which implies that the circles  $D(0, \|a_2\|), D(a_1 + a_2, \|a_2\|)$  cover  $R_A$ . It is now easy to visualize that since  $\gamma \notin \{a_1, a_2\}$ , we have either  $\|\gamma\| < \|a_2\|$  or  $\|\gamma - (a_1 + a_2)\| < \|a_2\|$ . This is a contradiction, since either  $\gamma$  or  $\gamma - a_1$  respectively would have been a shorter choice of vector than  $a_2$ .

## 1.4 Irreducible Lattices

In an attempt to say something more about the congruence of lattices we introduce the concept of reducible and irreducible lattices. This will help us to prove proposition 1.4.8 and the first inheritance theorem, both of which naturally seem intuitive, but are simultaneously hard to show rigorously. Their importance will be made clear in section 2.4.

**Definition 1.4.1** (Reducible & Irreducible Lattices).

i) We say that a lattice  $\Gamma$  is reducible if it is congruent to a lattice of the form  $\Gamma_1 \times \Gamma_2$  where  $\Gamma_1, \Gamma_2$  are of dimension at least 1. We say that  $\Gamma$  reduces into  $\Gamma_1 \times \Gamma_2$ .

ii) A lattice  $\Gamma$  is irreducible if it is not reducible.

We note that a lattice is irreducible if and only if its dual is irreducible. This comes from the discussion at the end of section 1.2.

**Lemma 1.4.2.** *Let  $A = [a_j] \in \mathbb{R}^{n \times n}$  be a matrix. Elements of  $B \in GL_n(\mathbb{Z})$  can by right multiplication change the order of the columns of  $A$  in any way. Elements of  $C \in O_n(\mathbb{R})$  can by left multiplication change the order of the rows of  $A$  in any way.*

*Proof.* Consider  $B = [e_{\sigma(j)}] = (\delta_{i\sigma(j)})_{ij}$  for some  $\sigma \in S_n$ , and write  $A = (a_{ij})_{ij}$ . We have  $AB = (\sum_k a_{ik} \delta_{k\sigma(j)})_{ij} = (a_{i\sigma(j)})_{ij} = [a_{\sigma(j)}]$  which is precisely a re-arrangement of the columns of  $A$ . Now consider

similarly  $C = [e_{\sigma^{-1}(j)}] = (\delta_{i\sigma^{-1}(j)})_{ij}$ . Then  $C$  changes the order of the rows of  $A$  in the following way,

$$CA = \left( \sum_k \delta_{i\sigma^{-1}(k)} a_{kj} \right)_{ij} = (a_{\sigma(i)j})_{ij} = \begin{bmatrix} a_{\sigma(1)1} & \cdots & a_{\sigma(1)n} \\ \vdots & \ddots & \vdots \\ a_{\sigma(n)1} & \cdots & a_{\sigma(n)n} \end{bmatrix}. \quad \textcircled{\circ}$$

**Lemma 1.4.3.** *The product of lattices  $\Gamma_1 \times \cdots \times \Gamma_m$  is congruent to  $\Gamma_{\sigma(1)} \times \cdots \times \Gamma_{\sigma(m)}$  for any  $\sigma \in S_m$ .*

*Proof.* See lemma A.0.6. \textcircled{\circ}

Each lattice  $\Gamma$  can be reduced into a finite number of irreducible lattices. In other words, for some  $n \geq 1$ ,  $\Gamma$  is up to congruency equal to

$$\Gamma_1 \times \cdots \times \Gamma_n,$$

where each  $\Gamma_i$  is irreducible. Moving forward we will consider the Minkowski sum of two discrete additive subgroups of  $\mathbb{R}^n$ , say  $A, B$ , as  $A + B := \{a + b : a \in A, b \in B\}$ . It is trivial to check that for an element of  $O_n(\mathbb{R})$  we have  $C(A + B) = CA + CB$ . We further write  $A \cdot B := \{a \cdot b : a \in A, b \in B\}$ . Before we can move on, we must give a more general description of lattices.

**Definition 1.4.4** (General Lattices). *Let  $v_1, \dots, v_k \in \mathbb{R}^n$  be a set of linearly independent vectors. They define a general lattice in the following way,*

$$v_1\mathbb{Z} + \cdots + v_k\mathbb{Z}.$$

*We may also write either  $\langle v_1, \dots, v_k \rangle_{\mathbb{Z}}$ ,  $\text{Span}_{\mathbb{Z}}\{v_j\}$  or  $[v_j]\mathbb{Z}^k$  to denote the general lattice and we say that  $[v_j]$  is its basis matrix, and  $k$  its dimension.*

**Proposition 1.4.5.** *An additive subgroup  $\Gamma \in \mathbb{R}^n$  is discrete if and only if it is a general lattice.*

*Proof.* See [17, p. 24]. \textcircled{\circ}

**Lemma 1.4.6.** *Let  $G, S$  be two non-trivial discrete additive subgroups of  $\mathbb{R}^n$ . An  $n$ -dimensional lattice  $\Gamma$  is irreducible if and only if  $\Gamma = G + S$  implies  $G \cdot S \neq \{0\}$ .*

*Proof.* We prove the negation,  $\Gamma$  is reducible if and only if  $\Gamma = G + S$  where  $G \cdot S = \{0\}$ .

$\Rightarrow$ ) If  $\Gamma$  is reducible, then  $\Gamma = C(\Gamma_1 \times \Gamma_2)$  for lattices  $\Gamma_1, \Gamma_2$  of dimension at least 1. Note that  $G = C(\Gamma_1 \times \{0\})$  and  $S = C(\{0\} \times \Gamma_2)$  are discrete additive groups. Since  $C$  preserves orthogonality, we have  $G \cdot S = \{0\}$ .

$\Leftarrow$ ) Now assume  $\Gamma = G + S$  and  $G \cdot S = \{0\}$ . Consider an orthonormal basis  $g_1, \dots, g_r$  spanning the smallest vector space  $V$  such that  $G \subseteq V$ . Let  $C$  be the orthonormal transformation that takes  $g_1, \dots, g_r$  to  $e_1, \dots, e_r$ , the standard basis. We observe that if  $Pr_1 : \mathbb{R}^n \rightarrow \mathbb{R}^r$  is the projection onto the first  $r$  coordinates, then  $Pr_1(C(G + S)) = \Gamma_1$  for some  $r$ -dimensional full-rank lattice  $\Gamma_1$  by proposition 1.4.5. If we do the analogous procedure with  $Pr_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{n-r}$ , we find  $C\Gamma = C(G + S) = Pr_1(C(G + S)) \times Pr_2(C(G + S)) = \Gamma_1 \times \Gamma_2$  for some lattices  $\Gamma_1, \Gamma_2$ . \textcircled{\circ}

**Lemma 1.4.7** (Congruence of Irreducible Products). *Two products of irreducible lattices,*

$$\Gamma_1 \times \cdots \times \Gamma_r \ \& \ \Lambda_1 \times \cdots \times \Lambda_s,$$

*are congruent if and only if  $r = s$  and there is a  $\sigma \in S_r$  such that all the pairs  $\Gamma_i, \Lambda_{\sigma(i)}$  are congruent.*

In this proof we use the notation of the direct sum  $\oplus$  instead of the Minkowski sum. The only difference is that when we write  $A \oplus B$  require that  $A \cdot B = \{0\}$ .

*Proof.*

$\Leftarrow$ ) Lemma 1.4.3 says that we can assume that  $\sigma = id$ . Then if  $\Lambda_i = C_i\Gamma_i$ , we get  $\Lambda_1 \times \cdots \times \Lambda_r = C(\Gamma_1 \times \cdots \times \Gamma_r)$  where  $C$  is the block matrix consisting of  $C_1, \dots, C_r$  along the diagonal.

$\Rightarrow$ ) Let first  $\dim(\Gamma_i) = n_i, \dim(\Lambda_j) = n'_j$ . For some  $C$  we have

$$\Lambda_1 \times \cdots \times \Lambda_s = C(\Gamma_1 \times \cdots \times \Gamma_r).$$

To ease notations, we define for  $1 \leq k \leq r, 1 \leq l \leq s$ ,

$$\Gamma(k) := \underbrace{\{0\} \times \cdots \times \{0\}}_{\#k-1} \times \Gamma_k \times \underbrace{\{0\} \times \cdots \times \{0\}}_{\#r-k} \quad \& \quad \Lambda(l) := \underbrace{\{0\} \times \cdots \times \{0\}}_{\#l-1} \times \Lambda_l \times \underbrace{\{0\} \times \cdots \times \{0\}}_{\#s-l},$$

in the obvious way precisely such that  $\bigoplus_{k=1}^r \Gamma(k) = \Gamma_1 \times \cdots \times \Gamma_r$  and  $\bigoplus_{l=1}^s \Lambda(l) = \Lambda_1 \times \cdots \times \Lambda_s$ . Let  $V := C\Gamma(1) \subseteq \Lambda_1 \times \cdots \times \Lambda_s$ . We will show that

$$V \subseteq \Lambda(l), \tag{*}$$

for some  $1 \leq l \leq s$ . To do this, we note that if  $V \subseteq U$ , then  $V = V \cap U$  and look at,

$$V = \bigoplus_{i=1}^s \underbrace{V \cap \Lambda(i)}_{:=G_i}.$$

It is clear that  $G_i \cdot G_j = \{0\}$  as long as  $i \neq j$  since this holds for  $\Lambda(i) \cdot \Lambda(j)$ . Further,  $G_i$  are discrete subgroups of  $\mathbb{R}^n$  since both  $V$  and  $\Lambda(i)$  are. We consider

$$\Gamma(1) = C^T V = \bigoplus_{i=1}^s C^T G_i.$$

It follows that  $C^T G_i \in \Gamma_1 \times \underbrace{\{0\} \times \cdots \times \{0\}}_{\#r-1} = \Gamma(1)$  for each  $1 \leq i \leq s$ . If we let  $\text{Pr} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_1}$  denote the projection of the first  $n_1$  coordinates in the obvious way, we get

$$\Gamma_1 = \bigoplus_{i=1}^s \text{Pr}(C^T G_i).$$

As a consequence of lemma 1.4.6 and by the fact that  $\text{Pr}(C^T G_i) \cdot \text{Pr}(C^T G_j) = C^T(G_i) \cdot C^T(G_j) = G_i \cdot G_j$ , we have that all but one  $\text{Pr}(C^T G_i)$  is the trivial group, implying directly that for at most one  $i$ ,  $G_i \neq \{0\}$ . It follows that  $(*)$  is true and the intersection  $C^T \Lambda(l) = \Gamma(1)$  is non-empty for the corresponding  $\Lambda(l)$ . Using the same arguments as for  $(*)$ , we find  $C^T \Lambda(l) \subseteq \Gamma(1)$  for this  $l$ . It follows that  $C^T C\Gamma(1) \subseteq C^T \Lambda(l) \subseteq \Gamma(1)$ , meaning  $C\Gamma(1) = \Lambda(l)$  and up to congruency by lemma 1.42, we may assume  $l = 1$ . It is clear now that  $n_1 = \dim \Gamma_1 = \dim \Lambda_1$ , since if  $\Gamma_1$  has  $m$  linearly independent vectors, then  $\Lambda_1$  must also have it and the other way around. We now find  $C' \in O_{n_1}(\mathbb{R})$  such that  $\Lambda_1 = C'\Gamma_1$ . Let  $C_0$  be the upper left  $n_1 \times n_1$  block matrix of  $C$ , we write

$$C = \begin{bmatrix} C_0 & C_2 \\ C_1 & C_3 \end{bmatrix} \quad \& \quad \Lambda(1) = C\Gamma(1) = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} \Gamma_1 \Rightarrow \Lambda_1 = C_0 \Gamma_1.$$

Now take  $x \in \Lambda(1)$ , since  $x_i = 0$  for  $i > n_1$  we have for a basis matrix  $A_1$  of  $\Gamma_1$  that

$$0 = C_1 A_1 \Rightarrow C_1 = 0 \in \mathbb{R}^{n-n_1 \times n_1}.$$

Finally, since each vector of  $C$  is of length one and orthogonal, this must also be true for  $C_0$  which shows that  $C_0 \in O_{n_1}(\mathbb{R})$  which is what we needed. To finish the proof we continue with the same procedure for  $\Gamma_2, \Gamma_3, \dots$  and so on until the invertible matrix  $C$  has paired up lattices from the product. This way, each lattice can be paired up and by invertibility of  $C$ , this pairing is unique and we conclude  $r = s$ .  $\blacksquare$

We now prove two important results with help of the above lemma. They will for example be used to show proposition 2.4.4, which has a compelling connection to Schiemann's theorem.

**Proposition 1.4.8.** *Two lattices  $\Gamma, \Lambda$  are congruent if and only if  $\Gamma^n, \Lambda^n$  are congruent for some  $n \in \mathbb{Z}_{\geq 2}$ .*

*Proof.*

$\Rightarrow$ ) This part follows directly.

$\Leftarrow$ ) We can reduce  $\Gamma$  and  $\Lambda$  into irreducible lattices  $\Gamma_1 \times \cdots \times \Gamma_r$  and  $\Lambda_1 \times \cdots \times \Lambda_s$  up to congruency. We therefore get by lemma 1.4.3 that

$$\Gamma_1^n \times \cdots \times \Gamma_r^n \stackrel{\mathcal{C}}{\sim} \Lambda_1^n \times \cdots \times \Lambda_s^n.$$

By the congruence of irreducible products we have  $r = s$ . Consider some lattices  $\Gamma_{i_0}$  in the left hand side product. Let  $\{\Gamma_{i \in I}\}$  be such that  $i \in I$  if and only if  $\Gamma_{i_0} \stackrel{\mathcal{C}}{\sim} \Gamma_i$ . Let similarly  $\{\Lambda_{i \in J}\}$  be the set of all  $\Lambda_i$  that are congruent to  $\Gamma_{i_0}$ . If  $|I| \neq |J|$ , then it is a direct consequence that there are different numbers of lattices among the irreducible products of  $\Gamma, \Lambda$  that are congruent to  $\Gamma_{i_0}$ , which is a contradiction of lemma 1.4.7. It follows that  $|I| = |J|$ . We conclude that we can find a bijection  $\sigma \in S_r$  such that the pairs  $\Gamma_i, \Lambda_{\sigma(i)}$  are congruent. By the lemma 1.4.7,  $\Gamma, \Lambda$  are congruent.  $\odot$

**Theorem 1.4.9** (Inheritance Theorem I). *Congruency and non-congruency are preserved under products. In other words, let  $\Gamma$  and  $\Gamma'$  be lattices of the same dimension. For any lattice  $\Lambda$  we have  $\Gamma \stackrel{\mathcal{C}}{\sim} \Gamma'$  if and only if  $\Gamma \times \Lambda \stackrel{\mathcal{C}}{\sim} \Gamma' \times \Lambda$ .*

*Proof.*

$\Rightarrow$ ) This direction is easy.

$\Leftarrow$ ) We can reduce  $\Gamma, \Gamma', \Lambda$  into irreducible lattices  $\Gamma_1 \times \cdots \times \Gamma_r, \Gamma'_1 \times \cdots \times \Gamma'_s$  and  $\Lambda_1 \times \cdots \times \Lambda_t$  respectively. We have  $r = s$  by lemma 1.4.7. Consider for some  $\Gamma_{i_0}$  the following set,  $\{(\Gamma_i)_{i \in I}, (\Lambda_j)_{j \in I'}\}$  where  $i \in I$  if and only if  $\Gamma_{i_0} \stackrel{\mathcal{C}}{\sim} \Gamma_i$  and  $j \in I'$  if and only if  $\Gamma_{i_0} \stackrel{\mathcal{C}}{\sim} \Lambda_j$ . Define  $\{(\Gamma'_i)_{i \in J}, (\Lambda_j)_{j \in J'}\}$  similarly to be the set of irreducible components of  $\Gamma' \times \Lambda$  that are congruent to  $\Gamma_{i_0}$ . We have by lemma 1.4.7 that  $|I| + |I'| = |J| + |J'|$ . There are of course an equal number of elements from the decomposition of  $\Lambda_j$  in both sets. It follows that  $|I| = |I'|$  and that there is a bijection  $\sigma \in S_r$  such that the pairs  $\Gamma_i, \Gamma'_{\sigma(i)}$  are congruent. By lemma 1.4.7 this means that  $\Gamma, \Gamma'$  are congruent.  $\odot$

## Chapter 2

# The Eigenvalue Problem

The perspective of spectral geometry on flat tori has a rich history and we devote this chapter to explaining the basics of it, and the most important tools that has arisen from it. We deal constantly with the *eigenvalue problem*; the problem of finding all the eigenfunctions and eigenvalues of the Laplace operator on a flat torus, whose definition we now recall. A flat torus  $\mathbb{T}_\Gamma$  is the set of equivalence classes in  $\mathbb{R}^n$  under the relation  $\sim$ , where  $u \sim v$  if and only if  $v - u \in \Gamma$ . In other words,

**Definition 2.0.1** (Flat Torus,  $\mathbb{T}_\Gamma$ ). *An  $n$ -dimensional flat tori is the quotient space*

$$\mathbb{T} = \mathbb{R}^n / \Gamma = \{v + \Gamma : v \in \mathbb{R}^n\}$$

for some  $n$ -dimensional lattice  $\Gamma$ . We write  $\mathbb{T}_\Gamma$  to emphasize the lattice, and we call  $\Gamma$  the underlying lattice of the flat torus.

Inspired by a series of lectures given by Hendrik Lorentz at the university of Göttingen, David Hilbert's PhD student Hermann Weyl would later in 1912 publish the result that is today known as *Weyl's law*. A direct consequence of the theorem was that the dimension and volume is determined by the Laplace operator on any bounded domain for functions that are zero on the boundary of the domain. The reason why the Laplace operator was originally studied in this context was its connection to sound frequencies, which makes this subject even more appealing for those who are musically inclined. A natural question to follow the discoveries of Weyl is whether the Laplace operator also determines the shape, meaning all information, of a manifold. This question was brought to light by Mark Kac who in 1966 asked if one could hear the shape of a drum. It was already known that this wasn't true in all dimensions, but the question was finally laid to rest when Gordon, Webb and Wolpert published their article *One Cannot Hear The Shape of A Drum* [12] in 1992, proving that the answer is no even in 2 dimensions. In our thesis, the example of the drum will not be of great importance, but it will be discussed section 2.2. Instead we turn our attention to the spectral geometry on flat tori, the history of which was mentioned in section 0.3. Flat tori can be modelled as Riemannian manifolds on which the eigenvalue problem can be posed, and we now ask Nilsson's question "can one hear the shape of a flat torus?" [8]. We say that two flat tori are of the same shape if they are isometric as in the definition below, and we give a description of this property.

**Definition 2.0.2** (Isometry of Flat Tori). *We say that two flat tori are isometric if they are isometric in the Riemannian sense, viewing the flat tori as Riemannian manifolds. We write  $\mathbb{T}_1 \stackrel{\mathcal{C}}{\sim} \mathbb{T}_2$  to denote that  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are isometric.*

**Theorem 2.0.3** (The Relation Between Isometry and Congruency). *Two flat tori are isometric in the Riemannian sense if and only if their underlying matrices are congruent.*

A proof of theorem 2.0.3 is given in [8, ch. 3]. In the specific case of flat tori, the result is also true if we instead consider the perhaps more familiar isometry in the sense of the quotient metric on the torus  $\mathbb{T}_\Gamma$  given by  $d_\Gamma([a], [b]) := \min\{a - b + \gamma : \gamma \in \Gamma\}$ . In order to know if the Laplace operator determines the shape of flat tori, we must first solve the eigenvalue problem given in definition 2.0.4. We shall consider different settings for which we solve it, but they all involve the Laplace operator,

$$\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

When considering functions  $f : \mathbb{T}_\Gamma \rightarrow \mathbb{C}$  for some  $n$ -dimensional flat torus  $\mathbb{T}_\Gamma$ , we might equivalently consider functions from  $g : \mathbb{R}^n \rightarrow \mathbb{C}$  that are  $\Gamma$ -periodic, meaning  $g(x + \gamma) = g(x)$  for all  $x \in \mathbb{R}^n$  and  $\gamma \in \Gamma$ . Consider now a basis matrix  $A$  for some lattice. We look at the  $L^2$  function space,

$$L^2(R_A) := \left\{ f : R_A \rightarrow \mathbb{C} \mid f \text{ is measurable and } \int_{R_A} |f|^2 < \infty \right\}.$$

Integration is done with respect to the Lebesgue measure and functions that agree almost everywhere are identified. The vector space  $L^2(R_A)$  is equipped with the inner product  $\langle f, g \rangle := \int_{R_A} f \bar{g}$  which means that we can talk about orthonormal bases of  $L^2(R_A)$ . The eigenvalue problem for flat tori can now be stated as follows:

**Definition 2.0.4** (The Eigenvalue Problem for Flat Tori and  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$ ). *Let  $\mathbb{T}_\Gamma$  be an  $n$ -dimensional flat torus with  $\Gamma = AZ^n$ . The eigenvalue problem is then to find non-zero  $\Gamma$ -periodic functions  $f$  such that  $f|_{R_A} \in L^2(R_A)$  and eigenvalues  $\lambda \in \mathbb{C}$  such that*

$$-\Delta f = \lambda f$$

in the distributional sense. We define  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$  to be the set of pairs  $(\lambda, m)$  such that  $\lambda$  is an eigenvalue from the eigenvalue problem and  $0 \neq m = \dim E_\lambda$ , where  $E_\lambda$  denotes the eigenspace of  $\lambda$ .

In other words,  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$  is the solution to the eigenvalue problem. The minus sign before the Laplace operator is in some ways redundant, but we add it for the sake of notation later on. As we will see, the solutions will be independent of the choice of basis matrix  $A$  of the underlying lattice  $\Gamma$ .

## 2.1 The Periodic Conditions

Throughout this paper, it is the *periodic* conditions that we are truly interested in. In the next section we will look at two other conditions that are very relevant for partial differential equations, but that don't necessarily have much to do with the flat torus. The periodic case conditions are nevertheless the precise conditions in definition 2.0.4. We begin with the following observation,

$$-\Delta f = \lambda f \iff (\Delta + \lambda id)f = 0.$$

Now  $\Delta + \lambda id$  is an *elliptic* operator and as a consequence of the elliptic regularity theorem, it is also *hypoelliptic*. The interested reader is referred to [19, p. 214-215]. All we need to know going forward is that since 0 is a smooth function, the fact that  $\Delta + \lambda id$  is hypoelliptic means that  $f$  must be a smooth function too, if it should solve the differential equation above. With this in mind we can proceed to solve the eigenvalue problem. In order to find all eigenfunctions, we aim to find an orthonormal basis of smooth  $\Gamma$ -periodic functions in  $L^2(R_A)$ . First, we find the following,

**Lemma 2.1.1.** *The functions  $(e^{2\pi i \gamma^* T x})_{\gamma^* \in \Gamma^*}$  are  $\Gamma$ -periodic eigenfunctions of the Laplace operator with eigenvalues  $4\pi^2 \|\gamma^*\|^2$  for  $\gamma^* \in \Gamma^*$ .*

*Proof.* For each  $\gamma^*$ , it is clear that  $e^{2\pi i \gamma^* T x}$  is a smooth function of  $x$ . Further,  $e^{2\pi i \gamma^* T (x+\gamma)} = e^{2\pi i \gamma^* T x} e^{2\pi i \gamma^* T \gamma} = e^{2\pi i \gamma^* T x}$  for any  $\gamma \in \Gamma$ , by definition of elements  $\gamma^*$  in the dual lattice. The last of the 3 conditions follows from the fact that  $(e^{2\pi i \gamma^* T x})'_{x_i} = 2\pi i \gamma_i^* e^{2\pi i \gamma^* T x}$  and the previous calculation. Finally, a straightforward calculation shows

$$-\Delta e^{2\pi i \gamma^* T x} = 4\pi^2 \|\gamma^*\|^2 e^{2\pi i \gamma^* T x}. \quad \odot$$

**Theorem 2.1.2** (Orthogonal Basis in The Periodic Case). *Let  $\Gamma$  be a lattice for which  $A$  is a basis matrix. The functions*

$$\left\{ e^{2\pi i \gamma^* T x} \right\}_{\gamma^* \in \Gamma^*}$$

of  $x$  form an orthogonal basis for smooth  $\Gamma$ -periodic functions  $f$  with  $f|_{R_A} \in L^2(R_A)$ .



*Proof.* See [8, ch. 4].

◉

Theorem 2.1.2 implies that any eigenfunction is a linear combination of  $(e^{2\pi i \gamma^* T x})_{\gamma^* \in \Gamma^*}$  and it is not hard to check that such a combination only has finitely many terms. Especially, there are no other eigenvalues than  $4\pi^2 \|\gamma^*\|^2$  for  $\gamma^* \in \Gamma^*$  (this can be seen by writing functions on the form  $f = \sum \langle f, e^{2\pi i \gamma^* T x} \rangle e^{2\pi i \gamma^* T x}$ ) and the dimension of the corresponding eigenspace for say  $\lambda$  is precisely  $\#\{\gamma^* \in \Gamma^* : \lambda = \|\gamma^*\|^2\}$ . The dimensions of the eigenspaces, and therefore the multiplicities in the spectra, are finite as a consequence of proposition 1.2.1. In other words, the solution to the eigenvalue problem for a given flat torus  $\mathbb{T}_\Gamma$  is

$$\text{Spec}_\Delta(\mathbb{T}_\Gamma) = \{(4\pi^2 \lambda, m) : 0 \neq m = \#\{\gamma^* \in \Gamma^* : \lambda = \|\gamma^*\|^2\}\}.$$

## 2.2 The Dirichlet & Neumann Conditions

Previously in this report we mentioned the historical connotations between the spectral geometry of flat tori and drums. So what is a *drum*? Mathematically we may think of it as any reasonable geometric shape that we can construct in  $\mathbb{R}^n$ . Or more generally as a Riemannian manifold. From the perspective of a physicist, the frequencies of a drum are calculated as the eigenvalues of functions that are defined on it and that vanish along the boundary of the drum, which will correspond to the *Dirichlet* boundary conditions. In this section we are not dealing with flat tori. We leave out the condition that functions should be  $\Gamma$ -periodic, but the techniques will be very similar to those of section 2.1.

**Definition 2.2.1** (Dirichlet & Neumann Boundary Conditions). *Let  $\mathcal{M}$  be a Riemannian manifold with a smooth boundary. Consider a function  $f : \mathcal{M} \rightarrow \mathbb{C}$ .*

- i)  $f$  satisfies the Dirichlet boundary conditions if  $f|_{\partial\mathcal{M}} = 0$ ,*
- ii)  $f$  satisfies the Neumann boundary conditions if  $\nabla f \cdot n|_{\partial\mathcal{M}} = 0$ .*

The *Neumann* boundary conditions can be rephrased as saying that the normal derivative of  $f$  should be zero at the boundary of  $\mathcal{M}$ . With this in mind, we state the eigenvalue problem to be the search for non-zero functions  $f \in L^2(\mathcal{M})$  satisfying some conditions  $\mathfrak{C}$  and eigenvalues  $\lambda \in \mathbb{C}$  such that  $-\Delta f = \lambda f$  in the distributional sense. We have the corresponding definition of  $\text{Spec}_\Delta^\mathfrak{C}(\mathcal{M})$  as in definition 2.0.4, where  $\mathfrak{C}$  emphasizes the conditions. We argue similarly as for the periodic conditions to see that the eigenfunctions must be smooth.

We shall only concern ourselves with manifolds on the form  $\mathcal{M} = R_A$ , where  $A$  is an invertible diagonal matrix, viewing it as a manifold in the natural way. The reason why we make this restriction is that the general case is much harder to solve, since we cannot compute the eigenvalues explicitly. We proceed as in the previous section.

**Lemma 2.2.2.** *Let  $\mathcal{M} = R_A$ , where  $A = \text{diag}(a_i)$  for non-zero real numbers  $a_i$ , be viewed as a Riemannian manifold in the natural way.*

- i) The functions  $(\prod_{k=1}^n \sin(\pi m_k x_k / a_k))_{m_k \in \mathbb{Z} \setminus \{0\}}$  are eigenfunctions of the eigenvalue problem with respect to the Dirichlet boundary conditions with eigenvalues  $\sum_{k=1}^n \pi^2 m_k^2 / a_k^2$ .*
- ii) The functions  $(\prod_{k=1}^n \cos(\pi(m_k + \frac{1}{2})x_k / a_k))_{m_k \in \mathbb{Z}}$  are eigenfunctions of the eigenvalue problem with respect to the Neumann boundary conditions with eigenvalues  $\sum_{k=1}^n \pi^2(m_k^2 + \frac{1}{2}) / a_k^2$ .*

The proof of this lemma is only a string of tedious calculations as in lemma 2.5, so we leave up to the reader. We now give a quite elegant lemma that leads up to theorem 2.2.4 and that is an important tool for finding orthogonal bases.

**Lemma 2.2.3.** *If  $\{e_i(x)\}_{i \in \mathbb{Z}}$  is an orthogonal basis for the subset of smooth functions in  $L^2([0, 1])$ , where each  $e_i$  is non-zero almost everywhere, then*

$$\{e_{i_1}(x_1)e_{i_2}(x_2) \cdots e_{i_n}(x_n)\}_{i_j \in \mathbb{Z}}$$

*is an orthogonal basis of the smooth functions in  $L^2([0, 1]^n)$ .*

*Proof.* We do the proof when  $n = 2$ , since the other cases can be done analogously. It is direct to check that the functions  $e_i e_j(x_1, x_2) = e_i(x_1) e_j(x_2)$  for  $i, j \in \mathbb{Z}$  form an orthogonal set. Therefore it is enough to check that for any smooth function in  $L^2([0, 1]^2)$  is a linear combination of the functions  $e_i e_j$ . If we can prove that given such a function  $f \in L^2([0, 1]^2)$  that  $0 = \langle f, e_i e_j \rangle$  for all  $e_i e_j$  implies  $f = 0$ , then we are done. To see this, assume that for such an  $f$ ,  $\langle f, e_i e_j \rangle = \lambda_{ij}$ . Then  $\langle f - \sum_{i,j} \lambda_{ij} e_i e_j, e_i e_j \rangle = 0$  for all  $e_i e_j$ , implying  $f = \sum_{i,j} \lambda_{ij} e_i e_j$ .

So let's show that  $0 = \langle f, e_i e_j \rangle$  for all  $e_i e_j$  implies  $f = 0$ . If we fix  $x_2$ , then  $f(x_1, x_2) e_i(x_2) \in L^1([0, 1])$  as a function of  $x_1$  for any  $i$ . We get the following for any fixed  $j_0$ ,

$$0 = \langle f, e_i e_{j_0} \rangle = \langle f \overline{e_{j_0}}, e_i \rangle \Rightarrow f \overline{e_{j_0}} = 0,$$

by the assumed completeness of  $\{e_i(x_1)\}_{i \in \mathbb{Z}}$  in  $L^1([0, 1])$ . This means  $f = 0$  since  $f$  is smooth and  $e_i$  non-zero almost everywhere.  $\odot$

**Theorem 2.2.4** (Orthogonal Bases in The Dirichlet & Neumann Case). *Let  $R_A$ , where  $A = \text{diag}(a_i)$  for non-zero real numbers  $a_i$ , be viewed as a Riemannian manifold in the natural way.*

i) *The functions  $\{\prod_{k=1}^n \sin(\pi m_k x_k / a_k)\}_{m \in \mathbb{N}_1^n}$  of  $x$  form an orthogonal basis of the subspace of smooth functions in  $L^2(\mathcal{R}_A)$  that satisfy the Dirichlet boundary conditions.*

ii) *The functions  $\{\prod_{k=1}^n \cos(\pi(m_k + \frac{1}{2})x_k / a_k)\}_{m \in \mathbb{N}_0^n}$  of  $x$  form an orthogonal basis of the subspace of smooth functions in  $L^2(\mathcal{R}_A)$  that satisfy the Neumann boundary conditions.*

*Proof.* See [18, p. 78] for the case where  $n = 1$ . We get the theorem by rescaling and applying lemma 2.9.  $\odot$

Here,  $\mathbb{N}_1$  is the set of positive integers and  $\mathbb{N}_0$  is the set of non-negative integers. Just as in section 2.1, we draw the conclusion that the eigenvalues given in lemma 2.2.2 are the only ones. We can then write out the spectra as follows, where  $D$  and  $N$  denotes Dirichlet and Neumann conditions respectively,

$$\text{Spec}_\Delta^D(R_{\text{diag}(a_i)}) = \{(\pi^2 \lambda, m) : 0 \neq m = \#\{\gamma^* \in \text{diag}(1/a_i) \mathbb{N}_1^n : \lambda = \|\gamma^*\|^2\}\},$$

$$\text{Spec}_\Delta^N(R_{\text{diag}(a_i)}) = \{(\pi^2(\lambda + \frac{1}{2} \sum 1/a_i^2), m) : 0 \neq m = \#\{\gamma^* \in \text{diag}(1/a_i) \mathbb{N}_0^n : \lambda = \|\gamma^*\|^2\}\}.$$

We now leave the discussion about the Dirichlet and Neumann boundary conditions until section 3.2, and resume with flat tori and the periodic conditions.

## 2.3 Poisson's Magic

An instance of when analysis makes great contributions to otherwise completely discrete fields is exemplified in this section. We will interestingly find that the length spectra of lattices determine their dimension and volume. The fact that we can do all of this easily with Poisson summation is remarkable or even *magical*. Throughout this section we follow [8, ch. 4]. First, let us formally define what we mean by isospectral flat tori and spectral determination.

**Definition 2.3.1** (Isospectrality and Spectral Determination). *Two flat tori  $\mathbb{T}, \mathbb{T}'$  are said to be isospectral if their Laplace spectra are the same, meaning*

$$\text{Spec}_\Delta(\mathbb{T}) = \text{Spec}_\Delta(\mathbb{T}').$$

*We write  $\mathbb{T} \stackrel{I}{\sim} \mathbb{T}'$  to denote this. Further, a flat torus  $\mathbb{T}$  is spectrally determined if for any other flat torus  $\mathbb{T}'$ , we have that  $\mathbb{T} \stackrel{I}{\sim} \mathbb{T}'$  implies  $\mathbb{T} \stackrel{C}{\sim} \mathbb{T}'$ .*

Before we state the Poisson summation formula, we define the volume of a flat torus, and note that it is well-defined since if  $A' = CAB$  for some  $B \in GL_n(\mathbb{Z})$  and  $C \in O_n(\mathbb{R})$ , then  $|\det(A')| = |\det(A)|$ .

**Definition 2.3.2.** *We say that the volume of a flat torus  $\mathbb{T}_\Gamma$  is the volume of its fundamental domains. In other words, if  $\Gamma = AZ^n$  then,*

$$\text{vol}(\mathbb{T}_\Gamma) := |\det(A)|.$$

**Theorem 2.3.3** (Poisson Summation). *Let  $f \in L^1(\mathbb{R}^n)$  and let  $\Gamma = A\mathbb{Z}^n$ . For  $x \in \mathbb{R}^n$ , the sum*

$$\sum_{\gamma \in \Gamma} f(x + \gamma)$$

*converges absolutely in  $L^1(R_A)$ -norm, and if  $\hat{f}$  is the  $n$ -dimensional Fourier transform of  $f$ , then the sum has the following Fourier expansion,*

$$\sum_{\gamma \in \Gamma} f(x + \gamma) = \frac{1}{\text{vol}(\mathbb{T}_\Gamma)} \sum_{\gamma^* \in \Gamma^*} \hat{f}(\gamma^*) e^{2\pi i \gamma^{*T} x}.$$

We refer to [8, section 4.4] for a proof of this fact. That the sum in the statement converges absolutely in  $L^1(R_A)$ -norm means in this case only that

$$\int_{R_A} \left| \sum_{\gamma \in \Gamma} f(x + \gamma) \right| dx < \infty.$$

**Corollary 2.3.4.** *For an  $n$ -dimensional lattice  $\Gamma$  we have the following two equalities for  $t \in (0, \infty)$ ,*

$$\sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t} = \frac{\text{vol}(\mathbb{T}_\Gamma)}{(4\pi t)^{n/2}} \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2 / 4t} \quad (2.1)$$

$$\sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2 / 4t} = \frac{(4\pi t)^{n/2}}{\text{vol}(\mathbb{T}_\Gamma)} \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t} \quad (2.2)$$

*Proof.* We let  $x = 0$  and consider  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  by  $f(y) = e^{-k\|y\|^2}$  for some  $k > 0$ . The Fourier transform of  $f$  is well-known and is given by  $\hat{f}(\xi) = (\pi/k)^{n/2} e^{-\pi^2 \|\xi\|^2 / k}$ . Poisson summation gives

$$\sum_{\gamma \in \Gamma} e^{-k\|\gamma\|^2} = \frac{1}{\text{vol}(\mathbb{T}_\Gamma)} \sum_{\gamma^* \in \Gamma^*} (\pi/k)^{n/2} e^{-\pi^2 \|\gamma^*\|^2 / k}.$$

Finally letting  $k = 4\pi^2 t$ , we get the statement by swapping  $\Gamma$  for  $\Gamma^*$  and using that  $(\Gamma^*)^* = \Gamma$ . The second equality is a rewriting the first one.  $\circ$

There are two results that importantly follows from corollary 2.3.4. As we see, the volume and dimension show themselves as constants and so we may hope to abuse this, after all, the exponents of (2.1) are precisely the eigenvalues of  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$ . In order to do this we look more closely at the left hand side of (2.1) via the following definition,

**Definition 2.3.5** (Theta Function). *Let  $\Gamma$  be a lattice. Then we define its theta function on  $(0, \infty)$  by,*

$$\theta_\Gamma(t) := \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2 \|\gamma^*\|^2 t}.$$

*We say that  $\theta_\Gamma$  is the theta function associated to the flat torus  $\mathbb{T}_\Gamma$ .*

To see that the theta function absolutely converges as a sum on the interval  $(0, \infty)$ , we could refer to generalizations of Weyl's law. However, due to the fact that the theta function is absolutely convergent in  $L^1(R_A)$ -norm by theorem 2.3.3, it must be convergent almost everywhere. Since each term of the sum is positive and grows strictly monotonely, it follows that it must be convergent everywhere on  $(0, \infty)$ . We give another motivation for this in section 4.1.

**Theorem 2.3.6** (The Spectrum & The Theta Function). *Let  $\mathbb{T}_\Gamma^n$  be a flat tori and let  $\theta_\Gamma$  be its associated theta function. Then  $\theta_\Gamma$  determines the eigenvalues up to the correct multiplicity for  $\mathbb{T}_\Gamma^n$ .*

*Proof.* First observe that we can write

$$\theta_\Gamma(t) = 1 + \sum_{i=0}^{\infty} m_i e^{-\lambda_i t},$$

where  $\lambda_i > 0$  is an enumeration of the distinct non-zero eigenvalues of the Laplace spectrum in increasing order, and  $m_i \neq 0$  are their respective multiplicities. For some  $r \in \mathbb{R}$  we consider

$$e^{rt}(\theta_\Gamma(t) - 1) = \sum_{i=1}^{\infty} m_i e^{(r-\lambda_i)t} \xrightarrow{t \rightarrow \infty} \begin{cases} 0 & \text{if } r < \lambda_1, \\ m_1 & \text{if } r = \lambda_1, \\ \infty & \text{if } r > \lambda_1. \end{cases}$$

In this way, both  $\lambda_1$  and  $m_1$  are theoretically determined. Inductively,

$$e^{rt}(\theta_\Gamma(t) - \sum_{i=0}^{N-1} m_i e^{-\lambda_i t}) = \sum_{i=N}^{\infty} m_i e^{(r-\lambda_i)t} \xrightarrow{t \rightarrow \infty} \begin{cases} 0 & \text{if } r < \lambda_N, \\ m_N & \text{if } r = \lambda_N, \\ \infty & \text{if } r > \lambda_N. \end{cases}$$

Again, we have derived the first  $N$  eigenvalues and their multiplicities. Continuing, we have that  $\theta_\Gamma$  determines these two objects uniquely.  $\odot$

Two natural consequences now follow. As a direct combination of theorem 2.3.6 and the fact that the exponents in theta function are precisely the values in  $\text{Spec}_\Delta(\mathbb{T}_\Gamma)$  with multiplicities, we state the first corollary. The second corollary is remarkable in that the proof is quite simple and uses analysis to say something about discrete sets of values. It can be directly translated as a result of lattices and quadratic forms, but we don't know how to prove it using only those perspectives.

**Corollary 2.3.7.** *Two flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_\Lambda$  are isospectral if and only if*

$$\theta_\Gamma(t) = \theta_\Lambda(t) \text{ for all } t \in (0, \infty).$$

**Corollary 2.3.8.** *Two isospectral flat tori are of the same dimension and they share the same volume.*

*Proof.* Let  $\mathbb{T}_\Gamma, \mathbb{T}_{\Gamma'}$  be  $n$  respectively  $m$ -dimensional flat tori. By corollary 2.3.4 and corollary 2.3.7 we have

$$\frac{\text{vol}(\mathbb{T}_\Gamma)}{(4\pi t)^{n/2}} \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t} = \frac{\text{vol}(\mathbb{T}_{\Gamma'})}{(4\pi t)^{m/2}} \sum_{\gamma' \in \Gamma'} e^{-\|\gamma'\|^2/4t}$$

for all  $t \in (0, \infty)$ . Assuming  $m \geq n$ , we rewrite the equality as the following,

$$(4\pi t)^{(m-n)/2} \frac{\text{vol}(\mathbb{T}_\Gamma)}{\text{vol}(\mathbb{T}_{\Gamma'})} \sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t} = \sum_{\gamma' \in \Gamma'} e^{-\|\gamma'\|^2/4t}.$$

Now if we let  $t \rightarrow \infty$ , then both sums tend toward 1, because they are absolutely convergent and we can put the limit inside. We directly arrive at a contradiction if  $m - n \neq 0$ , since then the left hand side would tend to infinity while the right hand side goes to 1. With  $m = n$  in mind, letting  $t \rightarrow \infty$  now shows the following,

$$\frac{\text{vol}(\mathbb{T}_\Gamma)}{\text{vol}(\mathbb{T}_{\Gamma'})} = 1. \quad \odot$$

The spectrum of a flat torus, consisting of all the lengths of the underlying dual lattice with multiplicities, is maybe not so intuitive to work with. Luckily however, proposition 2.3.10 gives a new characterization of isospectrality, leading up to which we define the length spectrum.

**Definition 2.3.9** (Length Spectrum & Isospectral Lattices). *For a lattice  $\Gamma$ , we define its length spectrum in the following way,*

$$\mathcal{L}_\Gamma := \{(\lambda, m) : 0 \neq m = \#\{\gamma \in \Gamma : \lambda = \|\gamma\|\}\}.$$

*We say that two lattices  $\Gamma, \Lambda$  are isospectral if they share the same length spectra. We write  $\Gamma \stackrel{I}{\sim} \Lambda$  if and only if  $\Gamma$  and  $\Lambda$  are isospectral.*

We recall that the length spectrum is a closed, discrete set for any lattice due to proposition 1.2.1, and as a consequence of the same proposition,  $m$  is always finite. The length spectrum of a geometric object has in general a different meaning, but as Scott Wolpert writes, the lengths of closed geodesics of a flat tori  $\mathbb{T}_\Gamma$  are given as  $\|\gamma\|$  for  $\gamma \in \Gamma$  which motivates our use of the word [5]. For convenience, it is usually a good idea to look at the *squared length spectrum* of lattice defined by

$$\mathcal{L}_\Gamma^{sq} := \{(\lambda^2, m) : (\lambda, m) \in \mathcal{L}_\Gamma\}.$$

It is clear that two lattices have equal length spectra if and only if their squared length spectra are equal.

**Proposition 2.3.10.** *Two flat tori  $\mathbb{T}_\Gamma$  and  $\mathbb{T}_{\Gamma'}$  are isospectral if and only if  $\Gamma$  and  $\Gamma'$  are.*

*Proof.* Without doing the details, in case that  $\mathbb{T}_\Gamma \stackrel{I}{\sim} \mathbb{T}_{\Gamma'}$  or  $\Gamma \stackrel{I}{\sim} \Gamma'$  we have, respectively, due to corollary 2.3.4,

$$\sum_{\gamma \in \Gamma} e^{-\|\gamma\|^2/4t} = \sum_{\gamma' \in \Gamma'} e^{-\|\gamma'\|^2/4t} \text{ or } \sum_{\gamma^* \in \Gamma^*} e^{-4\pi^2\|\gamma^*\|^2 t} = \sum_{\gamma'^* \in \Gamma'^*} e^{-4\pi^2\|\gamma'^*\|^2 t}.$$

In either case we proceed as in the proof of theorem 2.3.6 to find that  $\mathcal{L}_\Gamma = \mathcal{L}_{\Gamma'}$  in the first case and  $\text{Spec}_\Delta(\mathbb{T}_\Gamma) = \text{Spec}_\Delta(\mathbb{T}_{\Gamma'})$  in the second.  $\odot$

The above proposition can be rephrased as  $\Gamma \stackrel{I}{\sim} \Lambda$  if and only if  $\Gamma^* \stackrel{I}{\sim} \Lambda^*$  for any lattices  $\Gamma, \Lambda$ . We end by stating three very simple, but nonetheless fundamental for a lot of our deductions in specifically chapter 3. We leave the details to the reader, by only noting that the last two are direct consequences of proposition 2.3.10.

**Lemma 2.3.11.** *If  $\Gamma_1$  and  $\Gamma_2$  are lattices and  $\Gamma = \Gamma_1 \times \Gamma_2$ , then  $\theta_\Gamma = \theta_{\Gamma_1}\theta_{\Gamma_2}$ .*

*Proof.* This proof is a direct consequence of what we saw in section 1.2, namely that  $(\Gamma_1 \times \Gamma_2)^* = \Gamma_1^* \times \Gamma_2^*$ , but also the fact that  $\|(\gamma_1^*, \gamma_2^*)\|^2 = \|\gamma_1^*\|^2 + \|\gamma_2^*\|^2$  for any element  $(\gamma_1^*, \gamma_2^*) \in \Gamma_1^* \times \Gamma_2^*$ . We deduce as follows.

$$\begin{aligned} \theta_{\Gamma_1 \times \Gamma_2}(t) &= \sum_{\gamma^* \in (\Gamma_1 \times \Gamma_2)^*} e^{-4\pi^2\|\gamma^*\|^2 t} = \sum_{(\gamma_1^*, \gamma_2^*) \in \Gamma_1^* \times \Gamma_2^*} e^{-4\pi^2\|\gamma_1^*\|^2 t} e^{-4\pi^2\|\gamma_2^*\|^2 t} \\ &= \left( \sum_{\gamma_1^* \in \Gamma_1^*} e^{-4\pi^2\|\gamma_1^*\|^2 t} \right) \left( \sum_{\gamma_2^* \in \Gamma_2^*} e^{-4\pi^2\|\gamma_2^*\|^2 t} \right) = \theta_{\Gamma_1}(t)\theta_{\Gamma_2}(t). \end{aligned} \quad \odot$$

**Lemma 2.3.12.** *Congruency implies isospectrality both for flat tori and lattices.*

**Lemma 2.3.13.** *Two isospectral flat tori share the length of their respective shortest non-zero vectors.*

## 2.4 A Lower Bound on $N_n$

There are a lot of questions that naturally arise due to Schiemann's theorem. One of those has to do with the grows of  $N_n$ , which we define below and to which we give a lower bound in proposition 2.4.4,

**Definition 2.4.1** ( $N_n$ ). *We define the sequence  $N_n$  to be, for each dimension  $n$ , the maximal number of  $n$ -dimensional non-isometric flat tori that share a common Laplace spectrum. If there is no such maximal number for dimension  $n$ , we write  $N_n = \infty$ .*

We can now restate Schiemann's theorem as follows:  $N_1 = N_2 = N_3 = 1$  and  $N_4 \geq 2$ . We only know that  $N_n$  is finite if  $n < 3$ , and we conjecture that it is always finite. However, given a fixed spectrum, it is known that there can only be a finite number of flat tori that share it. This is discussed in section 3.4. We mention that  $N_n$  can be equivalently formulated in terms of lattice geometry or positive definite forms. To say something more about  $N_n$ , we start by giving a simple proof for the second inheritance theorem, which is just like the first theorem, only it deals with isospectrality instead of congruency.

**Theorem 2.4.2** (Inheritance Theorem II). *Isospectrality and non-isospectrality are preserved under products. In other words,  $\Gamma_1 \stackrel{I}{\sim} \Gamma_2$  if and only if  $\Gamma_1 \times \Gamma \stackrel{I}{\sim} \Gamma_2 \times \Gamma$  for any lattice  $\Gamma$ .*

*Proof.* Consider  $n$ -dimensional lattices  $\Gamma_1, \Gamma_2$  and an  $m$ -dimensional lattice  $\Gamma$ . Now  $\Gamma_1 \times \Gamma, \Gamma_2 \times \Gamma$  are isospectral if and only if  $\theta_{\Gamma_1 \times \Gamma} = \theta_{\Gamma_2 \times \Gamma}$  by corollary 2.3.7. This is equivalent to  $\theta_{\Gamma_1} \theta_{\Gamma} = \theta_{\Gamma_2} \theta_{\Gamma}$  by lemma 2.3.11. Since theta functions are non-zero where they are defined, this means exactly  $\theta_{\Gamma_1} = \theta_{\Gamma_2}$  which is true if and only if  $\Gamma_1, \Gamma_2$  are isospectral.  $\odot$

This result can also be proven by just looking at the length spectra instead, but using the theta function gives a more elegant argument. We now move on to the culmination of the inheritance theorems. Using both of them, we prove that since there are isospectral non-isometric flat tori in 4 dimensions, there are also such pairs in all higher dimensions. In other words, the following lemma is a crucial part of Schiemann's theorem.

**Lemma 2.4.3** (Schiemann's Lemma). *If there exists a sequence of  $\mathbb{T}_{\Lambda_1}, \dots, \mathbb{T}_{\Lambda_k}$  of mutually isospectral and pairwise non-isometric flat tori in dimension  $n$ , then there exists an equally long sequence of mutually isospectral and pairwise non-isometric flat tori in all higher dimensions.*

*Proof.* We do the proof in terms of the lattices. Consider any lattice  $\Gamma_m$  of dimension  $m \geq 1$ . By the inheritance theorems,  $\Lambda_1 \times \Gamma_m, \dots, \Lambda_k \times \Gamma_m$  still all share the same length spectrum and are all still pairwise non-congruent. Since these lattices are  $n + m$ -dimensional for any  $m$ , we are done.  $\odot$

It is possible to show Schiemann's lemma in a more elementary way. By induction, it is enough to do the proof for  $m = 1$ . For the sake of simplicity, let  $k = 2$ . With an appropriate choice of  $\lambda > 0$ , namely a  $\lambda$  that is not a length of at least one of the length spectra  $\mathcal{L}_{\Gamma}, \mathcal{L}_{\Lambda}$ , it is more or less direct to show that  $\Lambda_1 \times \lambda\mathbb{Z}, \Lambda_2 \times \lambda\mathbb{Z}$  are non-congruent. We end this section with the following observation in the form of a lower bound on  $N_n$ , which might already be known, but we have not seen it in any literature.

**Proposition 2.4.4.** *For each  $n \in \mathbb{N}$ , we have the following inequality,*

$$N_n \geq \left\lfloor \frac{n}{4} \right\rfloor + 1.$$

*In particular,  $N_n$  tends towards infinity as  $n$  does.*

*Proof.* By Schiemann's theorem,  $N_1 = N_2 = N_3 = 1$  and  $N_4 \geq 2$ . By Schiemann's lemma,  $N_n \geq 2$  as long as  $n \geq 4$ . In dimension  $4n$ , for some positive integer  $n$ , we claim that we can construct  $n + 1$  pairwise isospectral non-isometric flat tori. This would prove the statement by Schiemann's lemma. Now in 4-dimensions we have by Schiemann's theorem, two flat tori  $\mathbb{T}_{\Gamma}, \mathbb{T}_{\Lambda}$  that are isospectral and non-isometric. Consider the sequence of  $4n$ -dimensional lattices  $\Omega_i = \Gamma^i \times \Lambda^{n-i}$  for  $i = 0, \dots, n$ . As a direct consequence of the inheritance theorems and proposition 1.40, the  $n+1$  flat tori  $\mathbb{T}_{\Omega_i}$  all share a common Laplace spectrum, but are pairwise non-isometric.  $\odot$

# Chapter 3

## Some Special Cases

In this chapter, we have collected a number of relevant results. We start by proving Schiemann's theorem II, then we will show that rectangular flat tori are spectrally determined and say something about the Dirichlet and Neumann cases. On this note, we give a conjecture that if true, would generalize the rectangular case significantly. We end with notes on limits of flat tori.

### 3.1 Schiemann's Theorem II

The proof of the second part of Schiemann's theorem can be done using only elementary methods, and we leave this as an exercise for the reader. A more elegant proof uses results from previous chapters and may go as follows.

**Theorem 3.1.1** (Schiemann's Theorem II). *2-dimensional flat tori are spectrally determined.*

*Proof.* Let  $\mathbb{T}_\Gamma \stackrel{I}{\sim} \mathbb{T}_{\Gamma'}$ . Since congruency preserves isospectrality by lemma 2.3.12, we can by proposition 1.1.3 and theorem 1.3.7 assume that Minkowski reduced basis matrices  $A, A'$  for  $\Gamma, \Gamma'$  respectively are given by

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \quad \& \quad A' = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}.$$

Here, the diagonal entries are positive,  $(a, 0), (a', 0)$  are the shortest non-zero vectors of  $\Gamma, \Gamma'$  respectively and  $(c, b), (c', b')$  are the shortest vectors non-parallel to  $(a, 0), (a', 0)$  respectively. By lemma 2.3.13,  $a = a'$  and by corollary 2.3.9,  $b = b'$ . Now assume by contradiction that  $|c| < |c'|$ . Since  $(c', b')$  is the shortest vector in  $\Gamma'$  non-parallel to  $(a', 0)$  we find that the closed ball  $\overline{D}(0, \|(c, b)\|)$  contains more points from  $\Gamma$  than from  $\Gamma'$ , implying that they can't have the same length spectra. Therefore  $|c| = |c'|$  and if  $c = c'$  we are done. If not, then  $c = -c'$  and we have instead

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a' & c' \\ 0 & b' \end{bmatrix}.$$

In other words,  $\Gamma, \Gamma'$  are congruent by lemma 1.2.7 and  $\mathbb{T}_\Gamma \stackrel{\mathcal{C}}{\sim} \mathbb{T}_{\Gamma'}$  by theorem 2.0.3. ◻

### 3.2 Rectangular Flat Tori

In section 2.2 we indirectly touched on rectangular lattices, those with a diagonal basis matrix. We will now formally define this concept, and later prove that two rectangular lattices that are isospectral are also congruent. We'll end by saying something about the Dirichlet and the Neumann cases.

**Definition 3.2.1** (Rectangular Lattices). *A rectangular lattice  $\Gamma$  is a lattice with a diagonal basis matrix.*

Similarly we say that a flat torus is *rectangular* if its underlying lattice is. In order to prove theorem 3.2.3, we must make use of the following intuitive lemma, with the help of which, the proof will be elegant. The theorems of this section might already be known, but we have not been able to find any reference.

**Lemma 3.2.2.** *Let  $A = \text{diag}(d_i)$  be a basis matrix for a lattice  $\Gamma$ . Let  $\tilde{A} = \text{diag}(\chi_i d_{\sigma(i)})$  for some  $\sigma \in S_n$  be a reordering of the diagonal elements where  $\chi_i \in \{-1, 1\}$  may change the signs. The lattices  $\Gamma = A\mathbb{Z}^n, \tilde{\Gamma} = \tilde{A}\mathbb{Z}^n$  are isospectral and congruent.*

*Proof.* By lemma 1.4.2 we have that  $\text{diag}(d_{\sigma(i)}) = C \text{diag}(d_i) B$  for some  $B \in GL_n(\mathbb{Z})$  and  $C \in O_n(\mathbb{R})$ . Further, letting  $B' = [\chi_j e_j] \in GL_n(\mathbb{Z})$  we have  $\text{diag}(d_{\sigma(i)}) B' = \text{diag}(\chi_i d_{\sigma(i)})$ . We are done according to lemma 1.2.7.  $\odot$

**Theorem 3.2.3** (The Rectangular Case). *If two rectangular lattices are isospectral, then they are congruent.*

*Proof.* Let  $\mathbb{T}_\Gamma, \mathbb{T}_{\Gamma'}$  be  $n$ -dimensional isospectral flat tori with  $\Gamma = \text{diag}(d_i)\mathbb{Z}^n, \Gamma' = \text{diag}(d'_i)\mathbb{Z}^n$  for some non-zero real numbers  $d_i, d'_i$ . We can choose a reordering and change of signs  $\tilde{\Gamma} = \text{diag}(\chi_i d_{\sigma(i)})\mathbb{Z}^n, \tilde{\Gamma}' = \text{diag}(\chi'_i d'_{\tau(i)})\mathbb{Z}^n$  such that

$$0 < \chi_1 d_{\sigma(1)} \leq \dots \leq \chi_n d_{\sigma(n)} \ \& \ 0 < \chi'_1 d'_{\tau(1)} \leq \dots \leq \chi'_n d'_{\tau(n)}.$$

By lemma 3.2.2 and lemma 2.3.12  $\Gamma$  and  $\tilde{\Gamma}$  are congruent and isospectral. The same is true for  $\Gamma'$  and  $\tilde{\Gamma}'$ . Therefore, if we can show that  $\tilde{\Gamma}$  and  $\tilde{\Gamma}'$  are congruent, then the same is true for  $\Gamma$  and  $\Gamma'$ . For this reason we assume that for  $\Gamma, \Gamma'$ ,

$$0 < d_1 \leq \dots \leq d_n \ \& \ 0 < d'_1 \leq \dots \leq d'_n.$$

Now we can write  $\Gamma = d_1\mathbb{Z} \times \dots \times d_n\mathbb{Z}, \Gamma' = d'_1\mathbb{Z} \times \dots \times d'_n\mathbb{Z}$ . Since  $\mathcal{L}_\Gamma^{sq} = \{k_1^2 d_1 + \dots + k_n^2 d_n : k_i \in \mathbb{Z}\}$ . It is clear that  $d_1$  is the shortest length of  $\Gamma$ , and similarly  $d'_1$  is the shortest length of  $\Gamma'$ . By lemma 2.3.13,  $d_1 = d'_1$ . The second inheritance theorem says that  $\Gamma_0 = d_2\mathbb{Z} \times \dots \times d_n\mathbb{Z}$  and  $\Gamma'_0 = d'_2\mathbb{Z} \times \dots \times d'_n\mathbb{Z}$  are isospectral. Repeating the same argument as before,  $d_2 = d'_2$ . Continuing this procedure we find  $d_i = d'_i$  so that  $\Gamma = \Gamma'$ , which completes the proof.  $\odot$

For this theorem we present a proof using tools from chapter 2, but we note that this we can also give an elementary proof simply looking at the values and multiplicities of the length spectra. On the note of rectangular lattices, it is only fitting that we prove the corresponding result for the different boundary conditions.

**Theorem 3.2.4** (The Dirichlet & Neumann Case). *Rectangular lattices are spectrally determined with respect to the Dirichlet boundary conditions and the Neumann boundary conditions. In other words, if  $\text{Spec}_\Delta^{\mathfrak{C}}(R_A) = \text{Spec}_\Delta^{\mathfrak{C}}(R_{A'})$  for diagonal matrices  $A, A'$ , then  $A\mathbb{Z}^n = A'\mathbb{Z}^n$ .*

*Proof.* Let  $\Gamma = A\mathbb{Z}^n$ , where  $A = \text{diag}(a_i)$  and all  $a_i$  are non-zero. By lemma 3.2.2, we may assume that  $a_i > 0$ . Let us first make the following definitions,

$$\begin{aligned} \theta_\Gamma^D(t) &:= \sum_{(\lambda, m) \in \text{Spec}_\Delta^D(R_A)} m e^{-\lambda t} = \sum_{\gamma^* \in \text{diag}(1/a_i)\mathbb{N}_1^n} e^{-\pi^2 \|\gamma^*\|^2 t}, \\ \theta_\Gamma^P(t) &:= \sum_{(\lambda, m) \in \text{Spec}_\Delta^P(R_A)} m e^{-\lambda t} = e^{-\frac{\pi^2}{2} \sum 1/a_i^2 t} \sum_{\gamma^* \in \text{diag}(1/a_i)\mathbb{N}_0^n} e^{-\pi^2 \|\gamma^*\|^2 t}. \end{aligned}$$

The fact that  $\theta_\Gamma^D(t), \theta_\Gamma^P(t)$  are well-defined on  $(0, \infty)$  is inherited from the theta function  $\theta_\Gamma(t)$ . Let  $\Gamma' = \text{diag}(a'_i)\mathbb{Z}^n$  be a different diagonal matrix, also with  $a'_i > 0$ , and let  $\mathfrak{C}$  denote either  $D$  or  $P$ . By the same argument as in the proof of theorem 2.3.6,  $\theta_\Gamma^{\mathfrak{C}}(t) = \theta_{\Gamma'}^{\mathfrak{C}}(t)$  if and only if  $\text{Spec}_\Delta^{\mathfrak{C}}(R_A) = \text{Spec}_\Delta^{\mathfrak{C}}(R_{A'})$ . Further, using the same method as in the proof of lemma 2.3.11, we have  $\theta_{\Gamma_1 \times \Gamma_2}^{\mathfrak{C}} = \theta_{\Gamma_1}^{\mathfrak{C}} \theta_{\Gamma_2}^{\mathfrak{C}}$ . Therefore, similarly as in the proof of 3.2.3, we may argue by induction and hence it is enough to show  $a_i = a'_j$  for some  $i, j$ .

**The Dirichlet case:** The smallest value of  $\lambda$  such that  $(\lambda, m) \in \text{Spec}_\Delta^D(R_A)$  is equal to  $\pi^2 \sum m_i^2/a_i^2$  where each  $m_i = 1$ . The second smallest is equal to  $\pi^2 \sum m_i^2/a_i^2$  where  $m_{i_0} = 2$  for some  $i_0$  and  $m_i = 1$  for  $i \neq i_0$ . If  $\text{Spec}_\Delta^D(R_A) = \text{Spec}_\Delta^D(R_{A'})$ , then the spectra must have the second shortest length incommon. It follows that  $3/a_i^2 = 3/a_j^2$  for some  $i, j$  implying that  $a_i = a_j$ .

**The Neumann case:** The smallest value of  $\lambda$  such that  $(\lambda, m) \in \text{Spec}_\Delta^P(R_A)$  is equal to  $\frac{\pi^2}{2} \sum 1/a_i^2$ . The second smallest is equal to  $\frac{\pi^2}{2} \sum 1/a_i^2 + \pi^2/a_i^2$  for some  $a_i$ . If  $\text{Spec}_\Delta^P(R_A) = \text{Spec}_\Delta^P(R_{A'})$ , then the spectra must have the second shortest length incommon. It follows that  $\pi^2/a_i^2 = \pi^2/a_j^2$  for some  $i, j$  implying that  $a_i = a_j$ .  $\odot$



### 3.3 A Conjecture

With the knowledge of theorem 3.2.3, we might ask the question whether a rectangular flat torus can possibly be isospectral to a flat torus that is not rectangular. We believe that the answer is no and that we can say even more, which we express with the following conjecture.

**Conjecture 3.3.1** (Decomposition into Products). *Let  $\Gamma_1, \Gamma_2$  and  $\Gamma$  be lattices. If  $\Gamma_1 \times \Gamma_2$  is isospectral to  $\Gamma$  and if the dimension of  $\Gamma_1$  is 3 or lower, then  $\Gamma$  is congruent to some product of lattices  $\Lambda_1 \times \Lambda_2$  where each  $\Lambda_1$  is congruent to  $\Gamma_1$  and  $\Lambda_2$  is isospectral to  $\Gamma_2$ .*

One of the implications of the conjecture would be that a lattice with a diagonal block basis matrix of block sizes 3 or lower can only be isospectral to lattices that are congruent to them. The reason why we believe this to be true is because of Schiemann's theorem. Since lattices that are of dimension 3 or lower are determined by their length spectra, it is intuitive that this would show itself when such lattices are part of some product.

We had originally wondered if the conjecture could always be true. However, it is enough to look at the examples of isospectral non-isometric flat tori in 12 dimensions to see that this is not the case. We give an explicit example showing that the statement of the conjecture doesn't hold in greater generality. First we need to define the following lattices,

$$D_n := \{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i \in 2\mathbb{Z}\} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ -1 & -1 & \cdots & -1 & 1 \end{bmatrix} \mathbb{Z}^n,$$

$$E_n := \{x \in \mathbb{Z}^n \dot{\cup} (\mathbb{Z} + \frac{1}{2}\mathbb{1})^n : \sum_{i=1}^n x_i \in 2\mathbb{Z}\} = \begin{bmatrix} 1/2 & 0 & \cdots & 0 & 0 \\ 1/2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1/2 & 0 & \cdots & 1 & 1 \\ 1/2 & -1 & \cdots & -1 & 1 \end{bmatrix} \mathbb{Z}^n,$$

where  $E_n$  is only defined for  $n \in 4\mathbb{N}_1$ . As we have previously mentioned, it was Kneser who found the first 12-dimensional example of isospectral non-isometric flat tori. The example he gave was the pair of  $\mathbb{T}_{D_{12}^*}$  and  $\mathbb{T}_{(E_8 \times D_4)^*}$ . The lattice  $D_n$  is irreducible as long as  $n > 2$ , which one can verify by looking at the corresponding root system and Dynkin diagram. Therefore  $D_n^*$  is also irreducible as explained at the beginning of section 1.4. So even though the product  $E_8^* \times D_4^*$  is isospectral to  $D_{12}^*$ ,  $D_{12}^*$  is irreducible.

### 3.4 Limits of Flat Tori

In this section we give some results about limits of flat tori and we, inspired by Kneser's original arguments [5], give a hopefully more direct proof that for a fixed Laplace spectrum, there can only be finitely many non-congruent flat tori that share it.

**Proposition 3.4.1.** *Let  $\mathbb{T}_s$  be a continuous family of isospectral tori defined for  $s \in [0, 1]$ . The tori  $\mathbb{T}_s$  are then congruent.*

By a *continuous family* we mean that  $\mathbb{T}_s$  has a continuous family of basis matrices  $A(s)$  such that  $\mathbb{T}_s = A(s)\mathbb{Z}^n$  with  $A(s)$  being continuous with respect to the Euclidean matrix norm. The simple proof of this statement was given by Scott Wolpert in 1978 [5]. I give a slightly altered result that is more fitted to our purposes:

**Lemma 3.4.2.** *Consider for  $k \in \mathbb{N}_1$  the sequence  $\Gamma_k = A(k)\mathbb{Z}^n$ . Assume that all  $\Gamma_k$  are mutually isospectral and that  $A(k) \rightarrow A$ . At some point the sequence  $\Gamma_k$  becomes stationary up to congruency.*

*Proof.* The positive definite quadratic forms  $A(k)^T A(k)$  all have the same image over  $\mathbb{Z}^n$ . It is clear that  $A$  is invertible, since  $|\det(A(k))|$  is constant by corollary 2.3.8, and by continuity of the determinant we must have  $|\det(A(k))| = |\det(A)| \neq 0$ . Therefore,  $\Gamma = A\mathbb{Z}^n$  is a lattice and the set of its lengths is a discrete set. Take any  $x \in \mathbb{Z}^n$  and consider by the triangle inequality  $\left| \|A(k)x\| - \|Ax\| \right| \leq \|(A(k) - A)x\|$ . For big  $k$ ,  $\|A(k)x\| = \|Ax\|$  since the spectra are discrete and constant. There is now a  $k$  big enough such that for all  $x = e_i + e_j$ , we have

$$x^T \left( A(k)^T A(k) - A^T A \right) x = 0.$$

By lemma 1.1.7,  $A(k)^T A(k) = A^T A$  for all big  $k$  which implies precisely  $I = (AA(k)^{-1})^T AA(k)^{-1}$ , meaning  $A(k) = C(k)A$  where  $C(k)$  is a sequence in  $O_n(\mathbb{R})$ . We are done by lemma 1.2.7.  $\odot$

**Proposition 3.4.3.** *Let  $\Gamma = A\mathbb{Z}^n$  be any lattice. There is a constant  $r > 0$  depending only on  $A$  such that if an invertible matrix  $A'$  has  $\|A - A'\| < r$  and if  $\Gamma' = A'\mathbb{Z}^n$  is isospectral to  $\Gamma$ , then  $\Gamma'$  and  $\Gamma$  are also congruent.*

*Proof.* If the statement is false, then there is a sequence as in the assumption of lemma 3.4.2 only that it doesn't become stationary up to congruency. This contradicts the statement of that lemma.  $\odot$

It would certainly be interesting to be able to find a function  $r(A)$  to explicitly find the values of  $r$  in this proposition. We now state the most important theorem that we apply on limits of flat tori in theorem 3.4.5. According to Cassels this result “may be said to have completely transformed the subject” in the context of lattice theory [6, p. 136].

**Theorem 3.4.4** (Mahler's Compactness Theorem). *Let  $\Lambda_i$  be an infinite sequence of lattices of the same dimension, satisfying the following two conditions,*

- i) There exists a number  $K > 0$  such that  $\text{vol}(\Lambda_i) \leq K$  for all  $i$ ,*
- ii) There exists a number  $r > 0$  such that  $\inf_{0 \neq v \in \Lambda_i} \|v\| \geq r$  for all  $i$ .*

*There is then a subsequence  $\Lambda_{i_k}$  such that  $L_{i_k}$  converges to some lattice  $L$ .*

*Proof.* See [6, p. 137-139].  $\odot$

**Theorem 3.4.5** (Finiteness Theorem). *The total number of non-isometric flat tori with a given Laplace spectrum is finite.*

*Proof.* We do the proof in terms of lattices. Assume that we have an infinite sequence  $\Lambda_i$  of isospectral non-congruent lattices. By corollary 2.3.8 and lemma 2.3.13 we have that  $\text{vol}(\Lambda_i) \neq 0$  is constant for each  $i$  and each  $\Lambda_i$  share the length of their shortest non-zero vector. The assumptions of Mahler's compactness theorem are therefore satisfied. We find a converging subsequence of bases  $\Lambda_{i_k} \rightarrow \Lambda$ . By lemma 3.4.2 we get a contradiction.  $\odot$

We end with yet another result of limits of flat tori. We believe that it adds to the understanding of this section. It is crucial that we require the limits  $A, L$  to be invertible.

**Theorem 3.4.6** (The Limit Theorem). *Let  $\Gamma_i = A_i\mathbb{Z}^n, \Lambda_i = L_i\mathbb{Z}^n$  be lattices for each  $i \in \mathbb{N}$ . Assume also that  $A_i \rightarrow A$  and  $L_i \rightarrow L$  in the standard norm where  $A, L$  are invertible. If we write  $\Gamma = A\mathbb{Z}^n, \Lambda = L\mathbb{Z}^n$ , then the following hold:*

- i) If  $\Gamma_i, \Lambda_i$  are congruent for each  $i$ , then so are  $\Gamma, \Lambda$ .*
- ii) If  $\Gamma_i, \Lambda_i$  are isospectral for each  $i$ , then so are  $\Gamma, \Lambda$ .*

*Proof.*

*i)* We have by lemma 1.2.7 that  $L_i^{-1}C_iA_i = B_i$  for sequences  $C_i \in O_n(\mathbb{R}), B_i \in GL_n(\mathbb{Z})$ . Since  $O_n(\mathbb{R})$  is a compact topological space, we have a subsequence  $i_k$  such that  $C_{i_k}$  converges to some  $C \in O_n(\mathbb{R})$ . Then  $L_{i_k}^{-1}C_{i_k}A_{i_k}$  converges to  $L^{-1}CA$ . Finally, since  $GL_n(\mathbb{Z})$  is a discrete subgroup,  $L^{-1}CA \in GL_n(\mathbb{Z})$ . By lemma 1.2.7 we are done.

*ii)* If  $\Gamma_i, \Lambda_i$  are isospectral, then  $\theta_{\Gamma_i} = \theta_{\Lambda_i}$  for each  $i$ . Now

$$|\theta_{\Gamma} - \theta_{\Lambda}| \leq |\theta_{\Gamma} - \theta_{\Gamma_i}| + |\theta_{\Gamma_i} - \theta_{\Lambda_i}| + |\theta_{\Lambda_i} - \theta_{\Lambda}| = |\theta_{\Gamma} - \theta_{\Gamma_i}| + |\theta_{\Lambda_i} - \theta_{\Lambda}|.$$

Therefore it is enough to show  $|\theta_\Gamma - \theta_{\Gamma_i}| \rightarrow 0$  (for  $\Lambda$  the proof is analogous). Since  $\theta_\Gamma(t) = \sum_{\alpha \in \mathbb{Z}^n} e^{-4\pi^2 \|A^{-T}\alpha\|t}$ , we only need to show  $|e^{c\|A^{-T}\alpha\|} - e^{c\|A_i^{-T}\alpha\|}| \rightarrow 0$  for any constant  $c \in \mathbb{R}$  and all  $\alpha \in \mathbb{Z}$ . The exponential is a continuous function so it is enough to observe  $|\|A^{-T}\alpha\| - \|A_i^{-T}\alpha\|| \leq \|(A^{-T} - A_i^{-T})\alpha\| \rightarrow 0$  as  $i \rightarrow \infty$ .  $\heartsuit$

# Chapter 4

## Polyhedral Cones & Quadratic Forms

Throughout this and the next chapter we will focus on the number theoretical perspective of Schiemann's theorem in order to prove the third part of the theorem. This chapter is devoted to preparing for chapter 5 and to give information that is useful for some open problems explained in the final chapter. We begin by recalling what a quadratic form is.

**Definition 4.0.1** (Quadratic Form). *An  $n$ -dimensional quadratic form is a function  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  given by*

$$q(x) = x^T Q x,$$

where  $Q \in \mathbb{R}^{n \times n}$  is a matrix. We say that the quadratic form  $q$  is given by  $Q$ .

We can think of the quadratic form as a polynomial in  $n$  variables such that each term has degree 2 if we add up the degrees of all variables. Meaning that  $q(x, y) = x^2 + 2xy + y^2$  is quadratic while  $q(x, y, z) = z^3 - y^2 + x^2$  is not. As we have previously noted, if  $q$  is given by  $Q$  and  $q'$  is given by the symmetric matrix  $(Q^T + Q)/2$ , then  $q, q'$  are equal everywhere. We recall that a *positive definite* quadratic form  $q$  has  $q(x) > 0$  for  $0 \neq x \in \mathbb{R}^n$  and a *semi-positive* definite form  $q$  has  $q(x) \geq 0$  for  $x \in \mathbb{R}^n$ . We make use of this with the following definition,

**Definition 4.0.2** ( $\delta_n^+, \delta_n^0$ ). *We define  $\delta_n^+$  to be the set of symmetric positive definite quadratic forms of dimension  $n$ . Similarly we let  $\delta_n^0$  be the set of symmetric semi-positive definite quadratic forms of dimension  $n$ .*

### 4.1 Positive Definite Quadratic Forms

To give some background and understanding of the positive definite and semi-positive definite quadratic forms we devote this section to stating and proving a couple of elementary results about them.

**Lemma 4.1.1.** *A matrix  $Q$  has  $x^T Q x > 0$  for all non-zero  $x \in \mathbb{Z}^n$  if and only if it has  $x^T Q x > 0$  for all non-zero  $x \in \mathbb{Q}^n$ .*

*Proof.*

$\Leftarrow$ ) This follows directly.

$\Rightarrow$ ) To each  $x \in \mathbb{Q}^n$  there is a non-zero integer  $k$  such that  $kx \in \mathbb{Z}^n$ , meaning

$$k^2(x^T Q x) = (kx)^T Q kx > 0 \Rightarrow x^T Q x > 0. \quad \odot$$

**Lemma 4.1.2.** *Let  $Q$  be a matrix with  $x^T Q x > 0$  for all non-zero  $x \in \mathbb{Z}^n$ . Then  $Q \in \delta_n^0$ .*

*Proof.* By lemma 4.1.1, we have that  $x^T Q x > 0$  for all non-zero  $x \in \mathbb{Q}^n$ . Assume that  $x_0^T Q x_0 < 0$  for some  $x_0 \in \mathbb{R}^n$ . By continuity of polynomials, the quadratic form given by  $Q$  is also continuous which would imply that for all  $x \in \mathbb{Q}^n$  close enough to  $x_0$ , we would have  $x^T Q x < 0$  which is a contradiction.  $\odot$

**Theorem 4.1.3** (Spectral Theorem). *For symmetric, real matrices, there exists an orthonormal basis of eigenvectors and each eigenvalue is real.*

**Lemma 4.1.4.**

- i)  $Q \in \delta_n^0$  if and only if  $Q$  has only non-negative real eigenvalues,
- ii)  $Q \in \delta_n^+$  if and only if  $Q$  has only positive real eigenvalues.

*Proof.* In both cases by the spectral theorem, we have  $Q = M^T D M$  for some orthogonal matrix  $M$  and a diagonal matrix  $D$  of eigenvalues. Therefore, when considering  $x^T Q x = (Mx)^T D Mx$  for all  $x \in \mathbb{R}^n$ , we might as well rephrase the conditions as  $y^T D y \geq 0$  for  $y \in \mathbb{R}^n$  and  $y^T D y > 0$  for  $0 \neq y \in \mathbb{R}^n$  respectively, since  $M$  is invertible. However,

$$y^T D y = \sum_{i=1}^n \lambda_i y_i^2.$$

The statement follows directly. ⊙

Let us with this result give another motivation for why  $\theta_\Gamma(t)$  is convergent. Observe that  $\|A^{-T}\|^2 = x^T A^{-1} A^{-T} x$  is a positive definite quadratic form, say  $q$ . By lemma 4.1.4, the eigenvalues of  $A^{-1} A^{-T}$  are positive and by the spectral theorem we have  $A^{-1} A^{-T} = M^T D M$  for some  $M \in O_n(\mathbb{R})$ . We get  $q(x) = \sum \lambda_i (Mx)_i^2 \geq \lambda_{\min} \sum (Mx)_i^2$ , where  $\lambda_{\min}$  is the smallest eigenvalue. Observe that  $\sum (Mx)_i^2 = Mx \cdot Mx = x \cdot x$  by proposition 1.1.4. It follows that

$$\theta_\Gamma(t) \leq \sum_{x \in \mathbb{Z}^n} e^{-4\pi^2 \lambda_{\min} (x_1^2 + \dots + x_n^2)t} = \prod_{i=1}^n \sum_{x_i \in \mathbb{Z}} e^{-4\pi^2 \lambda_{\min} x_i^2 t},$$

and each term of the right hand side product is certainly finite. One could for example estimate them by integrals. We now move on with further descriptions of  $\delta_n^0$  and  $\delta_n^+$ .

**Lemma 4.1.5.** Any  $Q \in \delta_n^0$  can be written  $Q = E^T E$  for some  $E \in \mathbb{R}^{n \times n}$ .

*Proof.* By the spectral theorem,  $Q = M^T D M$  and by lemma 4.1.4, the elements of  $D$  are non-negative. It then makes sense to take the square root of  $D$ , by taking the root out of each element. Then we let  $E = \sqrt{D} M$  so that  $E^T E = Q$ . ⊙

**Proposition 4.1.6.** For a matrix  $Q \in \mathbb{R}^{n \times n}$  we have  $Q \in \delta_n^+$  if and only if  $x^T Q x > 0$  for all  $0 \neq x \in \mathbb{Z}^n$  and  $Q$  is invertible.

*Proof.*

$\Rightarrow$ ) Since if  $x^T Q x > 0$  for all  $0 \neq x \in \mathbb{R}^n$ , it also holds for  $0 \neq x \in \mathbb{Z}^n$ . Further,  $x^T(Qx) \neq 0$  for all  $0 \neq x \in \mathbb{R}^n$  implies that  $Q$  is invertible.

$\Leftarrow$ ) By lemma 4.1.2,  $Q \in \delta_n^0$  and by lemma 4.7 we can write  $Q = E^T E$ . Since  $Q$  is invertible,  $E$  is as well. It follows that  $x^T Q x = \|Ex\|^2 = 0$  if and only if  $x = 0$ . By definition we have  $Q \in \delta_n^+$ . ⊙

An example of a matrix  $Q$  that has  $x^T Q x > 0$  for all  $0 \neq x \in \mathbb{Z}^n$ , but that is not invertible can be constructed as follows,

$$Q = \begin{bmatrix} 1 & -\pi \\ -\pi & \pi^2 \end{bmatrix}.$$

We have  $x^T Q x = (x_1 - \pi x_2)^2 > 0$  as long as  $x \in \mathbb{Z}^2 \setminus \{0\}$ , but  $x^T Q x = (x_1 - \pi x_2)^2 = 0$  when  $(x_1, x_2) \in \text{Span}_{\mathbb{R}}\{(\pi, 1)\}$ . The matrix  $Q$  isn't invertible, also the values of  $x^T Q x$  tend to 0 for some sequence  $z_k \in \mathbb{Z}^n$ , which leads us to Kronecker's theorem and its application in lemma 4.1.8. On the note of the discreteness of the image of a quadratic form, we also give lemma 4.1.9 to be used in chapter 5.

**Theorem 4.1.7** (Kronecker's Theorem). Given a matrix  $A \in \mathbb{R}^{n \times m}$  and a vector  $\beta \in \mathbb{R}^n$ , the following are equivalent,

- i)  $\forall \epsilon > 0 \exists p \in \mathbb{Z}^m, q \in \mathbb{Z}^n : |(Aq - p - \beta)_i| < \epsilon$ ,
- ii)  $\forall r \in \mathbb{Z}^n : A^T r \in \mathbb{Z}^m$  we have  $\beta \cdot r \in \mathbb{Z}$ .

*Proof.* See [10]. ⊙

**Lemma 4.1.8.** Let  $q(x) = x^T Q x$  be a quadratic form whose image  $q(\mathbb{Z}^n)$  lies in  $\mathbb{R}_0^+$  and assume that there exists an  $r > 0$  such that  $q(x) \geq r$  as long as  $0 \neq x \in \mathbb{Z}^n$ . It follows that  $Q \in \delta_n^+$ .

*Proof.* By lemma 4.1.2 and 4.1.4 we can write  $Q = E^T E$ , implying that  $q(x) = \|Ex\|^2$ . By contradiction, assume that there is a non-zero  $x \in \mathbb{R}^n$  such that  $Ex = 0$ . Then for each  $t \in \mathbb{R}$  we have  $E(tx) = 0$ . By Kronecker's theorem we can find a sequence  $z_k \in \mathbb{Z}^n$  and  $t_k \in \mathbb{Z}$  such that  $\|z_k - t_k x\| \rightarrow 0$  (by setting  $A = x, \beta = 0$ ). This means that  $\|E(z_k - t_k x)\| \rightarrow 0$  by continuity of  $E$ . However  $\|E(z_k - t_k x)\|^2 = \|Ez_k\|^2 = q(z_k) \geq r > 0$  for all  $k$  which is a contradiction.  $\odot$

**Lemma 4.1.9.** For each  $q \in \delta_n^+$ , the set  $q(\mathbb{Z}^n)$  is a closed and discrete set.

*Proof.* Let  $q \in \delta_n^+$  be given by  $q(x) = x^T Q x$ . By Cholesky decomposition, there is an invertible matrix  $A$  such that  $Q = A^T A$ . It follows that  $q(\mathbb{Z}^n) = \{\|Ax\|^2 : x \in \mathbb{Z}^n\}$ , which is closed and discrete by proposition 1.2.1.  $\odot$

## 4.2 Polyhedra & Cones

We give an introduction to some theory about polyhedra and cones. When we later mention *polytopes*, we refer to bounded polyhedra. We begin by formalizing the following definitions:

**Definition 4.2.1** (Polyhedra). A  $k$ -dimensional polyhedron is a set of points in  $\mathbb{R}^n$  so that the smallest vector space that contains it is  $k$ -dimensional given explicitly by the following system of inequalities,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

for some  $a_{ij}, b_i \in \mathbb{R}$  and a positive integer  $m$ . We call the equalities  $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$  its supporting hyperplanes.

**Definition 4.2.2** (Convex Cone). A convex cone in  $\mathbb{R}^n$  is a set, say  $K$ , such that if  $x, y \in K$ , then  $x + y \in K$  and  $\lambda x \in K$  for each  $\lambda > 0$ .

**Definition 4.2.3** (Polyhedral Cones). Let  $A, B$  be sets of  $n$ -dimensional vectors. Then we define a polyhedral cone in the following way,

$$P_c(A, B) := \{x \in \mathbb{R}^n : a \cdot x \geq 0, b \cdot x > 0 \text{ for each } a \in A \text{ and } b \in B\}.$$

The dimension of  $P_c$  is the dimension of the smallest vector space containing it.

We typically realize a polyhedral cone as that set of points  $x \in \mathbb{R}^n$  that satisfy  $A'x \geq 0$  and  $B'x > 0$  where  $A'$  and  $B'$  are the matrices with the rows from  $A$  and  $B$ . It is more convenient however to theoretically work with sets instead of matrices since we might want the set  $A$  or  $B$  to be empty. We observe that the polyhedral cone is indeed a polyhedron. From now on we will simply say *convex polytope* when referring to definition 4.2.1. We move on to define what a  $j$ -face is, which is a concept we will make use of later.

**Definition 4.2.4** (Facets,  $j$ -Faces & Faces). A facet of a  $k$ -dimensional polyhedron or a polyhedral cone is a  $k - 1$ -dimensional intersection of some of the supporting hyperplanes and the polyhedron or the closure of the polyhedral cone. A  $j$ -face is similarly a  $j$ -dimensional intersection. A face is a  $j$ -face for some  $j$ .

**Lemma 4.2.5.** Let  $P_c(A, B) \subset \mathbb{R}^n$  and  $P_c(C, D) \subset \mathbb{R}^m$  be polyhedral cones. Their cartesian product is also a polyhedral cone. More specifically:

$$P_c(A, B) \times P_c(C, D) = P_c\left(A \times \{0\} \cup \{0\} \times C, B \times \{0\} \cup \{0\} \times D\right) \subseteq \mathbb{R}^{n+m}.$$

*Proof.* The equality follows from the following deduction,

$$\begin{aligned} P_c(A, B) \times P_c(C, D) &= \{(x, y) \in \mathbb{R}^{n+m} : a \cdot x \geq 0, c \cdot y \geq 0, b \cdot x > 0, d \cdot y > 0 \text{ for each } a \in A, c \in C, b \in B, d \in D\}, \\ &= \{(x, y) \in \mathbb{R}^{n+m} : [a \quad 0] \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, [0 \quad c] \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, [b \quad 0] \begin{bmatrix} x \\ y \end{bmatrix} > 0, \\ &\quad [0 \quad d] \begin{bmatrix} x \\ y \end{bmatrix} > 0 \text{ for each } a \in A, c \in C, b \in B, d \in D\}. \end{aligned} \quad \ominus$$

We end this part of the chapter by stating some elegant results that will be important in both section 4.4 and chapter 5.

**Definition 4.2.6** (Convex Hull). *The convex hull of a set  $V \subseteq \mathbb{R}^n$  is the smallest convex set that contains  $V$ . We denote it by  $\text{conv}(V)$ .*

**Proposition 4.2.7.** *The convex hull of a set of points is given as the set of all convex combinations of those points. More explicitly,*

$$\text{conv}(x_1, \dots, x_k) = \{\lambda_1 x_1 + \dots + \lambda_k x_k : \lambda_i \geq 0 \text{ and } \sum \lambda_i = 1\}.$$

*Proof.* See [20, p. 46]. \(\ominus\)

**Lemma 4.2.8.** *The intersection of two polyhedral cones is given by the following,*

$$P_c(A, B) \cap P_c(C, D) = P_c(A \cup C, B \cup D).$$

*Proof.* Follows from the definition of a polyhedral cone. \(\ominus\)

**Lemma 4.2.9.** *Let  $P_c(A, B)$  be a non-empty polyhedral cone and  $U$  some closed set in  $\mathbb{R}^n$ . Then the closure of  $P_c(A, B)$  is equal to  $P_c(A \cup B, \emptyset)$ , and  $P_c(A, B) \subseteq U$  if and only if  $P_c(A \cup B, \emptyset) \subseteq U$ .*

*Proof.* Fix an element  $x$  of  $P_c(A, B)$  and take any  $y \in P_c(A \cup B, \emptyset)$ . By definition of these sets, we have for each  $\epsilon > 0$  that  $A(\epsilon x + y) \geq 0$  and  $B(\epsilon x + y) > 0$ , meaning  $\epsilon x + y \in P_c(A, B)$ . Letting  $\epsilon \rightarrow 0$ , it is clear that each such  $y$  is a limit point of  $P_c(A, B)$ . It follows that  $\overline{P_c(A, B)} = P_c(A \cup B, \emptyset)$ . Finally, it is a well-known result in topology that a set is included in a closed set if and only if its closure is. \(\ominus\)

To check whether  $P_c(A, B)$  is empty one can check that  $P_c(A \cup B, \emptyset) \subseteq \cap_{b \in B} b^\perp$ . To see this, let  $x$  be such that  $a \cdot x, b \cdot x \geq 0$  for all  $a \in A, b \in B$ . If any such  $x$  lies in  $\cap_{b \in B} b^\perp$ , then there can be no  $x$  such that  $a \cdot x \geq 0, b \cdot x > 0$  for all  $a \in A, b \in B$ . This can be conveniently done in a computer program, and it will be discussed in more detail in section 4.4.

### 4.3 The Set of Minkowski Reduced Forms

We return to the Minkowski reduction of quadratic forms that we discussed in section 1.3. This time we will focus on the 3-dimensional case and say something about higher dimensions. We explain how to find the well-known, finite number of linear inequalities that define the subset of  $\delta_3^+$  that consists of Minkowski reduced forms.

**Definition 4.3.1** ( $\mathcal{M}_n$ ). *We define  $\mathcal{M}_n$  to be the set of  $n$ -dimensional symmetric positive definite quadratic forms that are Minkowski reduced.*

We can naturally embed  $n$ -dimensional symmetric quadratic forms  $q(x) = x^T Q x$  where  $Q = (q_{ij})_{ij}$  in  $\mathbb{R}^{n(n+1)/2}$ . In 3 dimensions we do it as follows,

$$q \mapsto (q_{11}, q_{22}, q_{33}, q_{12}, q_{13}, q_{23}).$$

With this in mind, we state the following lemma.

**Lemma 4.3.2.** *As long as  $n \leq 4$ , a symmetric form  $q$  is a Minkowski reduced positive definite form if and only if the following hold,*

- i)  $0 < q_{11} \leq q_{22} \leq \dots \leq q_{nn}$ ,
- ii)  $q(x) \geq q_{kk}$  for  $x \in \{-1, 0, 1\}^n$  with  $x_k = 1$  and  $x_{k+1} = \dots = x_n = 0$ .

The full proof of this statement is for given in [7, p. 257-258]. Many details of the proof are however left to the reader, which is why the interested reader should keep lemma 4.1.8 in mind. A consequence of this lemma is that  $\mathcal{M}_2 = P_c(A, B)$  where  $A$  and  $B$  are realized as matrices as follows,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 2 \end{bmatrix} \quad \& \quad B = [1 \quad 0 \quad 0].$$

What we are really interested in however is the set  $\mathcal{M}_3$ . It is a bit more involved, but again using lemma 4.22 get the following result.

**Theorem 4.3.3** ( $\mathcal{M}_3$  as a Polyhedral Cone). *The set of symmetric positive definite Minkowski reduced forms  $q(x) = x^T Q x$  in 3-dimensions is given by the following systems of inequalities,*

$$\left\{ \begin{array}{l} 0 < q_{11} \\ 0 \leq q_{22} - q_{11} \\ 0 \leq q_{33} - q_{22} \\ 0 \leq q_{11} - 2q_{12} \\ 0 \leq q_{11} + 2q_{12} \\ 0 \leq q_{11} + q_{22} + 2q_{12} - 2q_{13} - 2q_{23} \\ 0 \leq q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23} \end{array} \right. \quad \& \quad \left\{ \begin{array}{l} 0 \leq q_{11} + q_{22} - 2q_{12} + 2q_{13} - 2q_{23} \\ 0 \leq q_{11} + q_{22} + 2q_{12} + 2q_{13} + 2q_{23} \\ 0 \leq q_{11} - 2q_{13} \\ 0 \leq q_{11} + 2q_{13} \\ 0 \leq q_{22} - 2q_{23} \\ 0 \leq q_{22} + 2q_{23} \end{array} \right.$$

In other words,  $\mathcal{M}_3 = P_c(A, B)$ , where  $A$  and  $B$  are realized as the matrices given by

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & -2 & -2 \\ 1 & 1 & 0 & -2 & -2 & 2 \\ 1 & 1 & 0 & -2 & 2 & -2 \\ 1 & 1 & 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \& \quad B = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

*Proof.* By lemma 4.22 we only need  $0 < q_{11} \leq q_{22} \leq q_{33}$  and that  $q(x) \geq q(y)$  for  $x, y$  as in the following table,

Values of $x, y$ such that $q(x) \geq q(y)$			
$x_1$	$x_2$	$x_3$	$y$
1	0	0	$e_1$
-1, 0, 1	1	0	$e_2$
-1, 0, 1	-1, 0, 1	1	$e_3$

Out of these 13 conditions, 10 are non-redundant which are those we gave in the statement. ⊙

**Theorem 4.3.4** ( $\mathcal{M}_n$  as a Polyhedral Cone). *Out of the infinitely many conditions for a symmetric matrix  $Q$  to be in  $\mathcal{M}_n$  as given by definition 1.3.4, all but finitely many are non-redundant. In other words,  $\mathcal{M}_n$  is a polyhedral cone.*



*Proof.* See [7, p. 256-257].

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As will be explained in section 6.1, the 7-dimensional case is of special interest. For the reader who wants to see how  $\mathcal{M}_7$  is described as a polyhedral cone as in theorem 4.3.4, we refer to Tammela's list, that can be found in [21, p. 20] and [25].

## 4.4 Calculating Edges

In order to determine whether a polyhedral cone is included in another polyhedral cone, which will be important in chapter 5, one may calculate the edges of the first cone and check if they satisfy the condition to be included in the second. Here, the edges are the 1-faces of a polyhedral cone, often represented by the vector that spans it (that's how they are stored in the computer). If the edges of  $P_c(A, B)$  are  $k_1, \dots, k_r$ , then

$$\overline{P_c(A, B)} = \sum_{i=1}^r k_i \mathbb{R}_0^+,$$

which is stated in lemma B.0.3. Therefore, if each  $k_i \in P_c(C, \emptyset)$ , then  $P_c(A, B) \subseteq P_c(C, \emptyset)$ . It is more subtle to check whether  $P_c(A, B)$  is included in a set  $P_c(C, D)$  where  $D$  is non-empty, but we won't concern ourselves with those situations. No distinct edge is redundant, in other words the set of edges are  $\mathbb{R}_0^+$ -linearly independent, which is stated in lemma B.0.2, and which we define as follows.

**Definition 4.4.1** ( $\mathbb{R}_0^+$ -linear independence). *We say that a set of vectors  $v_1, \dots, v_k \in \mathbb{R}^n$  are  $\mathbb{R}_0^+$ -linearly independent if no vector  $v_i$  can be written as an  $\mathbb{R}_0^+$ -linear combination of the other vectors.*

The algorithm that we shall use in chapter 5 regularly calculates the edges of polyhedral cones for the reason described above. We will devote this section to explain how Schiemann did that in his thesis, and we follow theorem 4.4.7 in the file CalcEdges.jl in appendix D. Assume that we know the edges  $k_1, \dots, k_r$  of some polyhedral cone  $P_c(A, B)$ . What can we say about the edges of a polyhedral cone  $P_c(A \cup \{v\}, B) \subseteq \mathbb{R}^n$  for some vector  $v$ ? We start by giving two definitions and three results that are taken from appendix B where they are proven,

**Definition 4.4.2** ( $v^{\geq 0}$  &  $v^\perp$ ). *Let  $v \in \mathbb{R}^n$  be any vector. We define*

$$v^{\geq 0} := \{x \in \mathbb{R}^n : v \cdot x \geq 0\} \quad \& \quad v^\perp := \{x \in \mathbb{R}^n : v \cdot x = 0\}.$$

**Definition 4.4.3** (Centrally Anti-Symmetric). *A set  $X \subseteq \mathbb{R}^n$  is called centrally anti-symmetric if  $0 \neq y \in X$  implies  $-y \notin X$ .*

**Corollary 4.4.4.** *Let  $k_1 \neq k_2$  represent different edges of a polyhedral cone  $P_c(A, \emptyset)$ . Let  $\{a_1, \dots, a_r\} = \{a \in A : k_1 \subseteq a^\perp\}$  and  $\{a'_1, \dots, a'_s\} = \{a \in A : k_2 \subseteq a^\perp\}$ . The following are equivalent:*

- i)  $k_i \mathbb{R}_0^+ + k_j \mathbb{R}_0^+$  is a 2-face of  $P_c(A, \emptyset)$ ,
- ii)

$$\dim \bigcap_{a \in \{a_1, \dots, a_r\} \cap \{a'_1, \dots, a'_s\}} a^\perp = 2.$$

**Lemma 4.4.5.** *Let  $\dim P_c(A, \emptyset) = r, v \neq 0$  and  $a \in A$ . We have*

- 1)  $\dim(P_c(A, \emptyset) \cap a^\perp) < r - 1 \Rightarrow P_c(A \setminus \{a\}, \emptyset) = P_c(A, \emptyset)$ .
- 2)  $\exists x \in P_c(A, \emptyset) : x \cdot v > 0 \Rightarrow \dim(P_c(A \cup \{v\}, \emptyset)) = r$ .

**Lemma 4.4.6.** *Let  $P_c(A, \emptyset)$  be a centrally anti-symmetric polyhedral cone and let  $K$  be the set of its edges and  $L$  the set of its 2-faces. Let  $v \neq 0$ . We have for the set  $K'$  of edges of  $P_c(A \cup \{v\}, \emptyset)$ :*

$$\{k \mathbb{R}_0^+ \in K'\} = \{k \mathbb{R}_0^+ \in K : k \cdot v \geq 0\} \dot{\cup} \{F \cap v^\perp : F = k_1 \mathbb{R}_0^+ + k_2 \mathbb{R}_0^+ \in L \ \& \ k_1 \cdot v > 0, k_2 \cdot v < 0\}.$$

Observe that  $k_1, k_2$  in the right most set of lemma 4.4.6 represent two edges in  $K$ . We are now ready to give the following theorem, which we apply in our code. As we shall see, there are three distinct cases. Since  $P_c(A \cup B, B) = P_c(A, B)$ , we will always assume that  $B \subseteq A$ . If  $P_c(A, B) = \emptyset$ , there is nothing to update, therefore assume  $P_c(A, B) \neq \emptyset$  and note that by lemma 4.2.9 we have  $\overline{P_c(A, B)} = P_c(A, \emptyset)$ .

**Theorem 4.4.7** (Calculating edges). *Let  $P_c(A, B) \neq \emptyset$  be a centrally anti-symmetric polyhedral cone such that  $B \subseteq A$ . Let  $k_1, \dots, k_r$  be the edges of  $P_c(A, B)$ . For a non-zero vector  $v$ , we have*

**Case 1:**  $k_i \in v^{\geq 0}$  for each  $i$ .

The edges of  $P_c(A \cup \{v\}, B)$  are  $k_1, \dots, k_r$ .

**Case 2:**  $k_i \notin v^{\geq 0}$  for some  $i$  and each  $k_j$  has  $k_j \cdot v \leq 0$ .

Let  $k'_1, \dots, k'_l$  be those edges among  $k_1, \dots, k_r$  that lie in  $v^\perp$ . If there are no such  $k'_i$ , then  $P_c(A \cup \{v\}, B)$  is either empty or equal to  $\{0\}$ . The set  $P_c(A \cup \{v\}, B)$  is empty if and only if  $k := \sum k'_i$  has  $k \cdot b = 0$  for some  $b \in B$ . If it is non-empty and non-zero, then its edges are  $k'_1, \dots, k'_l$ .

**Case 3:**  $k_i \notin v^{\geq 0}$  for some  $i$  and some  $k_j$  has  $k_j \cdot v > 0$ .

The set  $P_c(A \cup \{v\}, B)$  is non-empty and its edges are calculated as those edges among  $k_1, \dots, k_r$  such that  $k_i \cdot v \geq 0$  and the elements of the set

$$\{F \cap v^\perp : F = k_1 \mathbb{R}_0^+ + k_2 \mathbb{R}_0^+ \text{ is a 2-face of } P_c(A, \emptyset) \text{ such that } k_1 \cdot v > 0, k_2 \cdot v < 0\}.$$

*Proof.*

**Case 1:** We have  $P_c(A, B) \subseteq \overline{P_c(A, B)} = \sum k_i \mathbb{R}_0^+ \subseteq v^{\geq 0}$ . This means that each element  $x$  of  $P_c(A, B)$  satisfies  $x \cdot v \geq 0$ . It follows that  $P_c(A, B) = P_c(A \cup \{v\}, B)$ .

**Case 2:** If  $k \cdot b = 0$  for some  $b \in B$ , then since  $k'_i \cdot b \geq 0$  for each  $i$  (recall  $B \subseteq A$ ), this must imply that  $k'_i \cdot b = 0$  for each  $i = 1, \dots, l$ . Because of  $P_c(A \cup \{v\}, B) \subseteq P_c(A \cup \{v\}, \emptyset) = \sum k'_i \mathbb{R}_0^+$ , any element of  $P_c(A \cup \{v\}, B)$  is an  $\mathbb{R}_0^+$ -linear combination of  $k'_i$ , with  $x \cdot b > 0$  for each  $b \in B$ . We cannot find such an  $x$  if  $k \cdot b = 0$  for some  $b \in B$ . On the other hand, if  $k \cdot b > 0$  for each  $b \in B$ , then  $k \in P_c(A \cup \{v\}, B) \neq \emptyset$ . To see that  $k'_1, \dots, k'_l$  really are the edges of a non-empty and non-zero  $P_c(A \cup \{v\}, B)$ , we refer to lemma 4.4.6 and the fact that there are no edges among  $k_1, \dots, k_r$  such that  $k_i \cdot v > 0$ .

**Case 3:** The set  $P_c(A \cup \{v\}, B)$  is non-empty since we have some  $k_j$  with  $k_j \cdot v > 0$  and by definition of an edge, there is a neighbourhood surrounding it containing a point  $x \in P_c(A, B)$  such that  $x \cdot v > 0$ . The edges of  $P_c(A \cup \{v\}, B)$  are by lemma 4.4.6 calculated as those  $k_1, \dots, k_r$  such that  $k_i \cdot v \geq 0$  and the elements of the set

$$\{F \cap v^\perp : F = k_1 \mathbb{R}_0^+ + k_2 \mathbb{R}_0^+ \text{ is a 2-face of } P_c(A, \emptyset) \text{ such that } k_1 \cdot v > 0, k_2 \cdot v < 0\}. \quad \odot$$

We make some practical remarks about how to make computations quicker and how to perform certain steps of theorem 4.4.7 with the computer. We give the next lemma in order to remove some redundant constraint of a polyhedral cone. First we note that to calculate the dimension of a polyhedral cone with edges  $k_1, \dots, k_r$ , it is enough to calculate the number of linearly independent vectors among  $k_i$ , which amounts to calculating the rank of a corresponding matrix.

**Lemma 4.4.8.** *Let  $P_c(A, \emptyset)$  be a polyhedral cone of dimension  $d$  and with edges  $k_1, \dots, k_r$ . We have  $P_c(A, \emptyset) = P_c(A', \emptyset)$  for*

$$A' := \{c \in A : \#\{k_i : k_i \in c^\perp\} \geq d - 1\}.$$

*Proof.* By lemma 4.4.5 1), we know that if  $\dim(P_c(A, \emptyset) \cap c^\perp) < d - 1$ , then  $P_c(A \setminus \{c\}, \emptyset) = P_c(A, \emptyset)$ . By theorem 4.4.7 case 2, the edges of  $P_c(A, \emptyset) \cap c^\perp$  are those of  $k_1, \dots, k_r$  that lie in  $c^\perp$ . We know therefore that  $P_c(A, \emptyset) \cap c^\perp$  can be at most  $d - 2$ -dimensional if less than  $d - 1$  edges lie in  $c^\perp$ . Therefore we can omit all  $c \in A$  such that less than  $d - 1$  edges are in  $c^\perp$ .  $\odot$

To calculate the set from theorem 4.4.7 case 3, we find all the pairs  $(k_i, k_j)$  such that  $k_i \cdot v > 0$  and  $k_j \cdot v < 0$ . Then we determine which pairs span a 2-side using corollary 4.4.4 and by calculating the number of columns minus the rank of the matrix consisting of all  $c \in A \cup \{v\}$  such that  $c \cdot k_i = 0, c \cdot k_j = 0$  as rows. It follows by lemma 4.4.5 2) that the dimension of  $P_c(A \cup \{v\}, B)$  is the same as that of  $P_c(A, B)$ . In both case 2 and 3 of theorem 4.4.7 we also remove redundant restraints as in lemma 4.4.8.

# Chapter 5

## Schiemann's Theorem III

In this chapter we show the following result, where we denote 3-dimensional quadratic forms as *ternary*,

**Theorem 5.0.1** (Schiemann's Theorem III). *Ternary positive definite quadratic forms are determined by their representation numbers.*

The proof was given originally by Schiemann in 1994 in his german PhD thesis [1] and two years later he wrote an english summary [2]. It needs the use of a computer and includes many technical details. We hope to make it more understandable by writing out the details that Schiemann left out. Before we begin the first part, we give an overview and some basic definitions that we will have great use for.

### 5.0 Part 0 – An Overview

We give this introductory part to give an overview of the steps that are involved of the proof of theorem 5.0.1. First we shall define some important notions. We recall the representation numbers from section 0.1,

**Definition 5.0.1** (Representation Numbers). *If  $q$  is a  $n$ -dimensional positive definite quadratic form, its representation numbers are defined as follows for  $t \in \mathbb{R}_0^+$  and some subset  $X \subseteq \mathbb{Z}^3$ ,*

$$\mathcal{R}(q, t) := \#\{x \in \mathbb{Z}^n : q(x) = t\} \quad \& \quad \mathcal{R}_X(q, t) := \#\{x \in X : q(x) = t\}.$$

Before we move on to outlining the different steps of the proof, we note that for any  $n$ -dimensional form  $q$  we have  $q(kx) = k^2q(x)$  for each  $x \in \mathbb{Z}^n$  and  $k \in \mathbb{Z}$ . Therefore the values of a ternary form  $q$  are completely determined by its values on the set  $\mathbb{Z}_*^3$ , defined as follows,

**Definition 5.0.2** ( $\mathbb{Z}_*^3$ ).

$$\mathbb{Z}_*^3 := \{x \in \mathbb{Z}^3 \setminus \{0\} : \text{GCD}(x_1, x_2, x_3) = 1 \text{ and the last non-zero coordinate is positive}\}$$

#### 5.0.1 Steps of The Proof

**Step 1:** Define  $V, D$  and  $\Delta$ .

We begin the proof by considering the set  $V$  of *sign reduced* forms, which contains a unique representative of each ternary quadratic form. We then define  $D \subseteq V \times V$  to be a set of pairs whose representation numbers are identical and  $\Delta$ , a set of identical pairs of quadratic forms. Theorem 5.0.1 holds if we can prove that  $D \subseteq \Delta$ .

**Step 2:** Define  $K(X, x_1, \dots, x_k)$ .

We let  $K(X, x_1, \dots, x_k)$  be a certain subset of the closure of  $V$  such that  $x_1, \dots, x_k$  are successively minimal vectors with respect to each of its elements. This set will be crucial for the next step.

**Step 3:** Define the sequence  $\mathcal{T}_i$  of coverings of  $D$ .

The sequence  $\mathcal{T}_i$  is defined with the help of  $K(X, x_1, \dots, x_k)$  to be coverings of  $D$  in the sense that for all  $i$  the union over all elements of  $\mathcal{T}_i$  covers  $D$ . Each such covering is a refinement of the previous one and we prove that

$$\bigcap_{i \in \mathbb{N}} \bigcup_{T \in \mathcal{T}_i} T \cap (V \times V) = D.$$

**Step 4:** Show that  $\mathcal{T}_i$  becomes stationary and that when it does, each of its elements is a subset of  $\Delta$ .

Using a computer algorithm we find that the sequence of coverings becomes stationary, in other words

$$\bigcup_{T \in \mathcal{T}_n} T \cap (V \times V) = D$$

for some  $n \in \mathbb{N}$ . We check that each  $T \in \mathcal{T}_i$  is a subset of  $\Delta$ . It follows that  $D \subseteq \Delta$ .

## 5.0.2 Notes on Schiemann's Papers

Schiemann's german PhD thesis and his english summary have clear differences. The summary includes less details, but presents the algorithm in a simpler way. We follow the summary since it is easier, but the reader should keep in mind that the thesis gives a potentially faster algorithm. All of the results in this chapter are due to Schiemann, but some were left without proof and some rather insignificant statements are incorrect. For example, the proofs for lemma 5.1.4 and proposition 5.1.9 were left to the reader and many other proofs he gives are not very detailed.

## 5.1 Part 1 – Sign Reduction

To start with, we'd like to classify all ternary positive definite quadratic forms. We do this by finding a domain that contains a unique representative of each element in  $\delta_3^+$ . By a representative of  $q$ , we mean an equivalent quadratic form  $q'$  satisfying the relation  $q \sim q' \Leftrightarrow q(x) = q'(Bx)$  for some  $B \in GL_3(\mathbb{Z})$  and all  $x \in \mathbb{Z}^3$ . Recall that any element of  $\delta_3^+$  is represented by a symmetric matrix, and we embed them in  $\mathbb{R}^6$  as in section 4.3.

### 5.1.1 The Set $V$ and The Edges of Its Closure

The sign reduced form which we now define is called *Vorzeichennormalform* by Schiemann in his thesis and simply *reduced form* in his summary, but we say *sign reduced* to emphasize that it is distinct from Minkowski reduction.

**Definition 5.1.1** (Sign Reduced Form). *A ternary positive definite form  $f$  is said to be in sign reduced form if*

- 1a)  $f$  is Minkowski reduced,
- 1b)  $f_{12} \geq 0, f_{13} \geq 0,$
- 1c)  $2f_{23} > -f_{22},$

and the following boundary conditions hold:

- 2a)  $f_{12} = 0 \implies f_{23} \geq 0,$
- 2b)  $f_{13} = 0 \implies f_{23} \geq 0,$
- 3a)  $f_{11} = f_{22} \implies |f_{23}| \leq f_{13},$
- 3b)  $f_{22} = f_{33} \implies f_{13} \leq f_{12},$
- 4a)  $f_{11} + f_{22} - 2f_{12} - 2f_{13} + 2f_{23} = 0 \implies f_{11} - 2f_{13} - f_{12} \leq 0,$
- 4b)  $2f_{12} = f_{11} \implies f_{13} \leq 2f_{23},$
- 4c)  $2f_{13} = f_{11} \implies f_{12} \leq 2f_{23},$
- 4d)  $2f_{23} = f_{22} \implies f_{12} \leq 2f_{13}.$

By a *boundary condition* in the context of polyhedral cones, we mean a condition whose assumption is satisfied on some part of its boundary, in this case on a facet of the polyhedral cone of forms satisfying 1a, b, c). We now define the set  $V$  and characterize its closure. Proposition 5.1.3 is important to motivate the algorithm in section 5.2.

**Definition 5.1.2** (The Set  $V$ ). *The set of all ternary symmetric positive definite sign reduced forms embedded in  $\mathbb{R}^6$  is written  $V$ .*

**Proposition 5.1.3.** *To each ternary positive definite quadratic form  $q$  there is a unique equivalent form in  $V$ .*

*Proof.* See proposition C.0.3. ⊙

**Lemma 5.1.4.** *The closure of  $V$  is precisely given by the following system of inequalities,*

$$\begin{cases} 0 \leq q_{11} \leq q_{22} \leq q_{33} \\ 0 \leq 2q_{12} \leq q_{11} \ \& \ 0 \leq 2q_{13} \leq q_{11} \\ -q_{22} \leq 2q_{23} \leq q_{22} \\ q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23} \geq 0 \end{cases} \quad (5.1)$$

*Proof.* It is not hard to see that the set of all forms satisfying 1a, b, c) in definition 5.1.1, is 6-dimensional and that the assumptions of the boundary conditions of definition 5.1.1 are satisfied on facets of this set (see theorem 4.3.3). It follows that the closure  $\bar{V}$  includes all the forms that lie on these facets regardless of whether they satisfy the boundary conditions. For a technical proof, see lemma C.0.4. ⊙

When we talk about the closure of  $V$  we refer to the standard topology on  $\mathbb{R}^6$ . Noting that  $\bar{V}$  is a polyhedral cone due to lemma 5.1.4 (and clearly a centrally anti-symmetric set), we find a set of generators  $M$  of it which we decompose into  $M = M_1 \cup M_2 \cup M_3$  accordingly.

$$\begin{aligned} M_1 &:= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, & M_2 &:= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} \right\} \quad \& \\ M_3 &:= \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & \pm 1 \\ 1 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \right\}. \end{aligned}$$

All elements of  $M_3$  are positive definite, which is shown toward the end of appendix C. Moreover, the elements of  $M_3$  are all Minkowski reduced, which is a consequence of theorem 4.3.3. When trying to find edges of  $\bar{V}$ , one might instead calculate them with the help of a computer. The Polymake library for the language Julia contains such tools for example.

**Proposition 5.1.5.** *The elements of  $M$  generate  $\bar{V}$  over  $\mathbb{R}_0^+$ . More precisely,*

$$\bar{V} = \left\{ \sum \lambda_i q_i : q_i \in M, \lambda_i \in \mathbb{R}_0^+ \right\}.$$

*Proof.* For any  $q \in \bar{V}$  we have  $0 \leq q_{11} \leq q_{22} \leq q_{33}$ . Say that  $q'_{22} = q_{22} - q_{11} \geq 0$  and  $q''_{33} = q_{33} - q_{22} \geq 0$ . We now define

$$q_1 = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{11} & \delta_1 q_{23} \\ q_{13} & \delta_1 q_{23} & q_{11} \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & q'_{22} & \delta_2 q_{23} \\ 0 & \delta_2 q_{23} & q'_{22} \end{bmatrix} \quad \& \quad q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q''_{33} \end{bmatrix},$$

where  $\delta_1 = 1, \delta_2 = 0$  if  $q'_{22} = 0$  and  $\delta_1 = q_{11}/q_{22}, \delta_2 = q'_{22}/q_{22}$  if  $q'_{22} > 0$ . It is not hard to see that each  $q_i \in \bar{V}$  and  $q_1 + q_2 + q_3 = q$ . Therefore it is enough to check that forms on the form  $q_i$  for some  $i$  are generated by elements of  $M$ .

**Case  $q_1$ :** Here we can assume  $q_{11} > 0$ , otherwise  $q_1$  must be zero everywhere. To look at forms with equal diagonal elements, we might consider  $V'_1 := \{q \in \bar{V} : q_{11} = q_{22} = q_{33} = 2\}$ . By lemma 5.1.4,

$$K := \{(q_{12}, q_{13}, q_{23}) : q \in V'_1\} = ([0, 1] \times [0, 1] \times [-1, 1]) \cap \{(a, b, c) : a + b - c \leq 2\}.$$

It is clear that  $K$  is defined by seven closed half-spaces. It is a closed, finite convex set in  $\mathbb{R}^3$  and it has the following vertices,

$$M'_3 := \{(0, 0, \pm 1), (0, 1, \pm 1), (1, 0, \pm 1), (1, 1, 0), (1, 1, 1)\}.$$

In particular,  $K = \text{conv} M'_3$ . By proposition 4.2.7,  $K = \{\sum \lambda_i q_i : \sum \lambda_i = 1, q_i \in M'_3, \lambda_i \in \mathbb{R}_0^+\}$  which implies  $V'_1 = \{\sum \lambda_i q_i : \sum \lambda_i = 1, q_i \in M_3, \lambda_i \in \mathbb{R}_0^+\}$ . It follows that forms on the form  $q_1$  are generated by  $M_3$ .

**Case  $q_2$ :** Here we can assume  $q'_{22} > 0$ , since otherwise  $q_2$  must be zero everywhere. We consider  $V'_2 := \{q \in \bar{V} : q_{11} = 0, q_{22} = q_{33} = 2\}$ . By lemma 5.1.4,  $K := \{(q_{12}, q_{13}, q_{23}) : q \in V'_2\} = \{0\} \times \{0\} \times [-1, 1]$ . The vertices of  $K$  are  $M'_2 := \{(0, 0, \pm 1)\}$ . In particular, if  $q, q'$  are the two distinct forms of  $M_2$ , then  $V'_2 = \{\lambda_1 q + \lambda_2 q' : \lambda_1, \lambda_2 \in \mathbb{R}_0^+, \lambda_1 + \lambda_2 = 1\}$ . In other words,  $q_2$  is generated by  $M_2$ .

**Case  $q_3$ :** Here we can assume  $q''_{33} > 0$ , since otherwise  $q_3$  is zero everywhere. To look at forms with the first two diagonal elements being zero, we might consider  $V'_3 := \{q \in \bar{V} : q_{11} = q_{22} = 0, q_{33} = 1\}$ . Lemma 5.1.4 says that  $q_{12}, q_{13}, q_{23} = 0$  for any  $q \in V'_3$ , meaning that  $q_3$  is generated by  $M_1$ .  $\odot$

The elements of  $M$  are furthermore the edges of  $\bar{V}$ . To see this, one may argue by definition. To each element of  $M$  we can find supporting hyperplanes from (5.1) such that when intersected with the system itself, the solution is the edge spanned by the element. This is somewhat tedious, but certainly not difficult. One can also use a pre-existing computer program for this step. We are now define  $D, \Delta$ , with which we can reformulate theorem 5.0.1 to say  $D \subseteq \Delta$ .

**Definition 5.1.6** (The Sets  $D$  &  $\Delta$ ). *We define*

$$D := \{(f, g) : f, g \in V \text{ and } \mathcal{R}(f, t) = \mathcal{R}(g, t) \ \forall t \in \mathbb{R}_0^+\},$$

$$\Delta := \{(f, f) : f \in \bar{V}\}.$$

### 5.1.2 The Sets $\text{MIN}(X)$ & $K(X, x_1, \dots, x_k)$

As we approach section 5.2, we must introduce two more definitions and show some results about them. First we introduce an order on  $\mathbb{Z}^3$  that will, in terms of the set of *minimal vectors* of  $X$ , written  $\text{MIN}(X)$ , describe the elements of some subset of  $\mathbb{Z}^3$  that are minimal in some sense. This order will be of great help for our algorithm, especially in light of the proposition 5.1.9.

**Definition 5.1.7** (The Relation  $\preceq$  and The Set  $\text{MIN}(X)$ ). *Let  $x, y \in \mathbb{Z}^3$ . Then we define the relations  $\preceq$ , as*

$$x \preceq y \Leftrightarrow f(x) \leq f(y) \ \forall f \in V.$$

*The relation  $x \preceq y$  should be read as  $x$  preceding  $y$  or  $y$  succeeding  $x$ . Further, let  $X \subseteq \mathbb{Z}^3$ . We define  $\text{MIN}(X)$  by*

$$\text{MIN}(X) := \{x \in X : \forall y \in X \setminus \{x\} : y \not\preceq x\}.$$

To calculate the set  $\text{MIN}(X)$  in application, we refer to section 5.4. We now move on to prove the most important properties of the relation  $\preceq$  and the set  $\text{MIN}(X)$ . First we give a simple lemma.

**Lemma 5.1.8.** *For each  $q \in M$ , the set  $q(\mathbb{Z}^3)$  is a discrete set. It follows that for  $q \in \bar{V}$ ,  $q(\mathbb{Z}^3)$  is a discrete set.*

*Proof.* Since elements of  $M_3$  are positive definite, they have the wanted property by lemma 4.1.9. Let  $q$  be the element of  $M_1$ . We have  $q(\mathbb{Z}^3) = q'(\mathbb{Z})$  where  $q'$  is given by the matrix (1). Clearly  $q'(\mathbb{Z})$  is discrete and closed. Finally, let  $q$  be in  $M_2$ . We have  $q(\mathbb{Z}^3) = q''(\mathbb{Z}^2)$  where  $q''$  corresponds to the lower right  $2 \times 2$  matrix of  $q$ . But those matrices are also positive definite, so we are done by lemma 4.1.9.  $\odot$

**Proposition 5.1.9.**

- i) *The relation  $\preceq$  is transitive,*
- ii)  *$x \preceq y \Leftrightarrow f(x) \leq f(y) \ \forall f \in M$ ,*
- iii)  *$x \preceq y$  implies  $\|x\| \leq \|y\|$  and  $|x_3| \leq |y_3|$ ,*
- iv) *If  $x \preceq y \preceq x$ , then  $x = \pm y$ ,*
- v)  *$\text{MIN}(X)$  is finite for each  $X \subseteq \mathbb{Z}^3$ ,*
- vi)  *$\text{MIN}(X)$  is non-empty for each non-empty  $X \subseteq \mathbb{Z}_*^3$ .*

*Proof.*

- i) If  $x \preceq y$  and  $y \preceq z$  then by definition  $f(x) \leq f(y) \leq f(z)$  for all  $f \in V$ . This implies  $x \preceq z$ .
- ii)

⇐) By  $V \subseteq \bar{V}$  we know that any  $f \in V$  is of the form  $f = \sum \lambda_i f_i, f_i \in M$  and  $\lambda_i \in \mathbb{R}_0^+$ . We get  $f(y) = y^T (\sum \lambda_i f_i) y = \sum \lambda_i f_i(y) \geq \sum \lambda_i f_i(x) = f(x)$ .

⇒) Take  $f_{i_0} \in M$ . Since  $f_{i_0} \in \bar{V}$ , there is a sequence  $f^j = \sum \lambda_i^j f_i \in V$  such that  $f^j \rightarrow f_{i_0}$ . By  $\mathbb{R}_0^+$ -linear independence we know that  $\lambda_{i_0}^j \rightarrow 1$  and  $\lambda_i^j \rightarrow 0$  for  $i \neq i_0$ . Now we have for each  $j$ ,

$$\sum \lambda_i^j f_i(y) \geq \sum \lambda_i^j f_i(x) \Rightarrow f_{i_0}(y) \geq f_{i_0}(x) + \frac{1}{\lambda_{i_0}^j} \sum_{i \neq i_0} \lambda_i^j (f_i(x) - f_i(y)).$$

Letting  $j \rightarrow \infty$  we have by the discussion above that for each  $\epsilon > 0$ ,  $f_{i_0}(y) \geq f_{i_0}(x) + \epsilon$ .

iii) Consider  $x = (x_1, x_2, x_3) \neq y = (y_1, y_2, y_3)$  with  $x \preceq y$ . We have that  $x_3^2 \leq y_3^2$  by ii) and the form in  $M_1$ . If we add the first and the second form in  $M_3$  we get  $f = (4, 4, 4, 0, 0, 0)$  with  $f(x) \leq f(y)$  so that  $x_1^2 + x_2^2 + x_3^2 \leq y_1^2 + y_2^2 + y_3^2$ .

iv) Looking at the following three forms generated by  $\bar{V}$ ,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

we find  $|x_i| = |y_i|$  for  $i = 3, 2, 1$  in that order. Further, the form with positive elements of  $M_2$  gives

$$x_2 x_3 \leq y_2 y_3 \leq x_2 x_3.$$

If  $x_3 \neq 0$  and  $x_3 = y_3$ , then  $x_2 = y_2$ . If  $x_3 = -y_3$ , then  $x_2 = -y_2$ . Using the same argument we finally get  $x = -y$ , when consider the following inequalities that we get by adding the fifth and sixth form of  $M_3$ ,

$$x_1 x_2 \leq y_1 y_2 \leq x_1 x_2.$$

v) Let  $X \subseteq \mathbb{Z}^3$ . Assume that we have a sequence of  $x^i = (x_1^i, x_2^i, x_3^i) \in \text{MIN}(X)$  for  $i = 1, 2, 3, \dots$  with  $x_i \not\preceq x_k$  for  $k > i$ . Then  $\|x^i\| \rightarrow \infty$  and we can choose a subsequence  $x^{i_j}$  starting at some point  $x^{i_0}$  such that each coordinate monotonely either increases or decreases, and each coordinate either is constant or goes to  $\pm\infty$ . We find the contradiction by looking at each  $f \in M$  and seeing that with this sequence  $x^{i_j}$ , for big enough  $j$  we have  $f(x^{i_j}) \geq f(x^{i_0})$ . This is evident for the element in  $M_1$ . For  $M_2$  forms we have  $f(x^{i_j}) = f(x^{i_0})$  if  $x_2^{i_j}, x_3^{i_j}$  are constant. Otherwise we have by discreteness of  $f(\mathbb{Z}^3)$  due to lemma 5.1.8, that  $f(x^{i_j}) \rightarrow \infty$ . For elements  $f \in M_3$ , we have that  $f(x^{i_j}) \rightarrow \infty$  by virtue of lemma 5.1.8 and that  $\|x^i\| \rightarrow \infty$ .

vi) Consider the shortest length of a vector in  $X$ . It is greater than 0, since  $0 \notin \mathbb{Z}_*^3$ , and denote all distinct vectors in  $X$  that share this shortest length by  $u_1, \dots, u_k \in X$ . The only way that  $\text{MIN}(X)$  can be empty is that for each  $u_i$  there is a  $x \neq u_i$  in  $X$  such that  $x \preceq u_i$ , but this  $x$  has to be equal to  $u_j$  for some  $j \neq i$  due to iii). If  $k = 1$ , then  $u_1 \in \text{MIN}(X)$  for the same reason. If  $k > 1$ , and if  $\text{MIN}(X)$  were empty, then we could find two vectors  $u_i \neq u_j$  such that  $u_j \preceq u_i \preceq u_j$  as a consequence of the set of  $u_i$  being finite. By iv), this implies  $u_j = \pm u_i$  which is a contradiction since they are distinct and both  $u_i, -u_i$  can't be in  $\mathbb{Z}_*^3$ .  $\odot$

In his thesis, Schiemann writes that any sequence  $x_1 \succeq x_2 \succeq x_3 \dots$  eventually becomes stationary. However, this is not true. Simply note that  $(0, 0, -1) \preceq (0, 0, 1) \preceq (0, 0, -1) \preceq (0, 0, 1) \preceq \dots$  does not become stationary. This means for example that  $\text{MIN}(\{(0, 0, -1), (0, 0, 1)\}) = \emptyset$ .

We continue by stating the definition of  $K(X, x_1, \dots, x_k)$ , which we may read as the  $K$ -set for a given  $X$  and a sequence of  $x_i$ , and we also give a description of it as a polyhedral cone. The significance of this the  $K$ -set is due to lemma 5.1.12, but cannot be fully understood until section 5.2.

**Definition 5.1.10** (The  $K$ -Set). *Let  $X \subseteq \mathbb{Z}_*^3$ , and let  $x_1, \dots, x_k \in X$  be distinct. We define,*

$$K(X, x_1, \dots, x_k) := \{f \in \bar{V} : f(x_i) = \min(f(X \setminus \{x_1, \dots, x_{i-1}\})) \forall i = 1, \dots, k\}.$$

For  $k = 0$ , we simply set  $K(X) := \bar{V}$ .

The  $K$ -set is the set of quadratic forms in  $\bar{V}$  for which  $x_1, \dots, x_k$  are the successively smallest vectors in  $X$ . We note the following direct connection to the representation numbers, which motivates our use of this definition: if  $f \in K(X, x_1, \dots, x_k, x)$ , then

$$f(x) = \min f(X \setminus \{x_1, \dots, x_k\}) = \min \left\{ t_0 \in \mathbb{R}_0^+ : \sum_{0 \leq t \leq t_0} \mathcal{R}_X(f, t) \geq k + 1 \right\}.$$

**Lemma 5.1.11.**  $K(X, x_1, \dots, x_k)$  is a polyhedral cone. More precisely,

$$K(X, x_1, \dots, x_k) = \{f \in \bar{V} : f(x_1) \leq \dots \leq f(x_k) \text{ and } f(x_k) \leq f(x) \quad \forall x \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})\}.$$

*Proof.* It follows by definition that  $K(X, x_1, \dots, x_k)$  is the set of  $f \in \bar{V}$  such that  $f(x_1) \leq \dots \leq f(x_k)$  and  $f(x_k) \leq f(x)$  for all  $x \in X \setminus \{x_1, \dots, x_k\}$ . Since  $\text{MIN}(X \setminus \{x_1, \dots, x_k\}) \subseteq X \setminus \{x_1, \dots, x_k\}$ , the “ $\subseteq$ ” inclusion follows. To see “ $\supseteq$ ”, assume that there exists a  $y \in X \setminus \{x_1, \dots, x_k\}$  such that  $y \notin \text{MIN}(X \setminus \{x_1, \dots, x_k\})$ . By definition, there is a  $y_1 \in X \setminus \{x_1, \dots, x_k, y\}$  such that  $y_1 \preceq y$ . Either  $y_1 \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})$  or we can find  $y_2 \in X \setminus \{x_1, \dots, x_k, y, y_1\}$  such that  $y_2 \preceq y_1$ . By proposition 5.1.9 *iii*), this sequence must end, meaning for some  $y_i$  we have  $y_i \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})$ . By transitivity,  $y_i \preceq y$ , meaning  $f(y_i) \leq f(y)$  for all  $f \in \bar{V}$ . And since  $f(x_k) \leq f(y_i)$ , we also have  $f(x_k) \leq f(y)$ .  $\odot$

To see how the  $K$ -set is realized as a polyhedral cone we proceed as follows. For fixed  $x, y \in \mathbb{R}^n$  and any symmetric quadratic form  $f$  we have the following,

$$f(y) \geq f(x) \Leftrightarrow \sum_i f_{ii}(y_i^2 - x_i^2) + \sum_{i < j} f_{ij}(2y_i y_j - 2x_i x_j) \geq 0.$$

The right hand side is a linear inequality for the values of  $f_{ij}$ . Given  $x_i$  and that we can calculate  $\text{MIN}(X \setminus \{x_1, \dots, x_k\})$ , by noting that  $\text{MIN}(X \setminus \{x_1, \dots, x_k\})$  is a finite set, we can explicitly write out all the linear inequalities defining the  $K$ -set to store in a computer.

**Lemma 5.1.12.** Let  $X \subseteq \mathbb{Z}_*^3$  and  $x_1, \dots, x_k \in X$  be distinct. If  $X \setminus \{x_1, \dots, x_k\} \neq \emptyset$ , then

$$K(X, x_1, \dots, x_k) = \bigcup_{y \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})} K(X, x_1, \dots, x_k, y).$$

*Proof.*

“ $\supseteq$ ”: This inclusion is trivial since  $K(X, x_1, \dots, x_k, y) \subseteq K(X, x_1, \dots, x_k)$  by definition.

“ $\subseteq$ ”: Let  $f \in K(X, x_1, \dots, x_k)$  and

$$Y_f = \left\{ y \in X \setminus \{x_1, \dots, x_k\} : f(y) = \min f(X \setminus \{x_1, \dots, x_k\}) \right\} \subseteq X \setminus \{x_1, \dots, x_k\}.$$

The set  $Y_f$  is non-empty by the discreteness of  $f(X \setminus \{x_1, \dots, x_k\})$  given by lemma 5.1.8. By proposition 5.1.9 *vi*),  $\text{MIN}(Y_f) \neq \emptyset$ . Fix some  $y_0 \in \text{MIN}(Y_f)$ . We will proceed to show that  $y_0 \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})$ . In that case, we are done since by definition we have  $f \in K(X, x_1, \dots, x_k, y_0)$ . First we consider the following decomposition,

$$X \setminus \{x_1, \dots, x_k\} = Y_f \dot{\cup} X \setminus \{x_1, \dots, x_k\} \setminus Y_f.$$

To see that  $y_0 \in \text{MIN}(X \setminus \{x_1, \dots, x_k\})$ , it suffices to show that  $x \not\preceq y_0$  for any  $y \neq x \in X \setminus \{x_1, \dots, x_k\}$ . Since  $y_0 \in \text{MIN}(Y_f)$ , we know that  $x \not\preceq y_0$  for  $y_0 \neq x \in Y_f$ . Now consider  $x \in X \setminus \{x_1, \dots, x_k\} \setminus Y_f$  and note that  $x \neq y_0$  (if  $X \setminus \{x_1, \dots, x_k\} \setminus Y_f = \emptyset$ , then we are already done by the decomposition). For such an  $x$ , we get  $f(x) > \min f(X \setminus \{x_1, \dots, x_k\})$ , since  $x \notin Y_f$ . This means  $f(x) > f(y_0)$ , since  $y_0 \in Y_f$  which implies  $x \not\preceq y_0$  and we are done.  $\odot$

## 5.2 Part 2 – Coverings of $D$

This is the part of the proof that uses polyhedral cones to define coverings of the set  $D$ . We will give explicit constructions of the coverings and towards the end show some of their properties.



### 5.2.1 Calculating $T_{V \times V}$

From now on we consider polyhedral cones  $T \subseteq \bar{V} \times \bar{V} \subseteq \mathbb{R}^{12}$ . By definition of  $V$ , we can write

$$V = P_c(\mathfrak{A}, \mathfrak{B}) \cap \{f \in \mathbb{R}^6 : \forall (c, d) \in \mathfrak{C} : (c \cdot f = 0 \Rightarrow d \cdot f \geq 0)\}.$$

Here,  $\mathfrak{A}, \mathfrak{B}$  are conditions corresponding to 1a), b) and to 1c) respectively in the definition of  $V$ .  $\mathfrak{C}$  corresponds in the obvious way to the boundary conditions of the sign reduced form and is realized as the variables Cbound and Dbound in the file VariDeclared.jl. Note that even though  $V$  is not a polyhedral cone, because of the boundary conditions,  $P_c(\mathfrak{A}, \mathfrak{B})$  is.

**Lemma 5.2.1.** *Let  $T = P_c(A, B) \subseteq \bar{V} \times \bar{V}$ . We define a sequence of polyhedral cones  $T_i$  for  $i \in \mathbb{N}_0$  by*

$$\begin{aligned} T_0 &:= \left( P_c(\mathfrak{A}, \mathfrak{B}) \times P_c(\mathfrak{A}, \mathfrak{B}) \right) \cap T, \\ T_i &:= T_{i-1} \cap \bigcap_{(c,d) \in \mathfrak{C}: T_{i-1} \subseteq (c,0)^\perp} \{(f, g) \in \mathbb{R}^6 \times \mathbb{R}^6 : f \cdot d \geq 0\} \cap \\ &\quad \cap \bigcap_{(c,d) \in \mathfrak{C}: T_{i-1} \subseteq (0,c)^\perp} \{(f, g) \in \mathbb{R}^6 \times \mathbb{R}^6 : g \cdot d \geq 0\}. \end{aligned}$$

The sequence  $T_i$  becomes stationary at some  $i_0$  and we write  $T_{V \times V} := T_{i_0}$ . Further,  $\overline{T \cap (V \times V)} = \overline{T_{V \times V}}$  and  $T \cap (V \times V) \subseteq T_{V \times V} \subseteq T$ .

For a proof of the equality, see lemma C.0.5. We will not use it for the proof of Schiemann's theorem, but it is theoretically interesting. Note that Schiemann's proof as stated in his PhD thesis seems to be incorrect.

*Proof.* Clearly we have  $T_{i+1} \subseteq T_i$  for each  $i$ . Let  $C_1^i, C_2^i$  be the subsets of all  $c$  such that  $(c, d) \in \mathfrak{C}$  with the property that  $T_i \subseteq (c, 0)^\perp, T_i \subseteq (0, c)^\perp$  respectively. Since  $T_i$  monotonely decreases,  $C_1^i$  and  $C_2^i$  monotonely increase. However  $\mathfrak{C}$  is finite meaning that  $C_1^i$  and  $C_2^i$  converge and become stationary. It follows that  $T_i$  must become stationary.

We are left to show  $T \cap (V \times V) \subseteq T_{V \times V}$ . Take any  $(f, g) \in T \cap (V \times V)$ . It is clear that  $(f, g) \in T_0$ . Now say  $(f, g) \in T_i$ . When calculating  $T_{i+1}$ , note that  $T_i \subseteq (c, 0)^\perp$  only if  $f \cdot c = 0$  at which point we already know  $f \cdot d \geq 0$  for  $(c, d) \in \mathfrak{C}$ . We have the analogous situation for  $T_i \subseteq (0, c)^\perp$ . In either case, this implies  $(f, g) \in T_{i+1}$ .  $\odot$

We end this subsection by stating the following lemma without proof; it comes directly from the procedure of calculating  $T_{V \times V}$  in lemma 5.2.1.

**Lemma 5.2.2.** *Let  $T = P_c(A, B) \subseteq \bar{V} \times \bar{V}$ . We have  $(T_{V \times V})_{V \times V} = T_{V \times V}$ .*

### 5.2.2 Defining The Sequence $\mathcal{T}_i$

We are ready to define the coverings that Schiemann gave of the set  $D$  from definition 5.1.6. We follow his summary. First we formalize two definitions.

**Definition 5.2.3** (Covering & Refinement). *A covering of some set  $S \subseteq \mathbb{R}^n$  is simply a set  $P$  of sets in  $\mathbb{R}^n$  such that*

$$S \subseteq \bigcup_{U \in P} U.$$

*A refinement of  $P$  is a covering  $P'$  such that*

$$S \subseteq \bigcup_{U' \in P'} U' \subseteq \bigcup_{U \in P} U.$$

The covering property that we now define should give you an idea of the data structure we use, but this will be explained in later sections. The reader who wonders how to do the steps that we define in this section using computers may rest assured that this will be discussed in later sections.

**Definition 5.2.4** (The Covering Property). *A covering  $\mathcal{T}$  of  $D$  has the covering property if each  $T \in \mathcal{T}$  has the covering property, meaning it satisfies the following.*

**P1:**  *$T$  is a polyhedral cone and  $T = T_{V \times V}$ ,*

**P2:** *The set  $\Lambda = \Lambda(T) \in \{\emptyset, \langle e_1, e_2 \rangle_{\mathbb{Z}}, \langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}\}$  is maximal with  $f|_{\Lambda(T)} = g|_{\Lambda(T)}$  for all  $(f, g) \in T$ . Further, there is a number  $k = k(T) \in \mathbb{N}_0$  and  $x_1, \dots, x_k, y_1, \dots, y_k \in \mathbb{Z}_*^3 \setminus \Lambda$  with*

$$\begin{aligned} T &\subseteq K(\mathbb{Z}_*^3 \setminus \Lambda, x_1, \dots, x_k) \times K(\mathbb{Z}_*^3 \setminus \Lambda, y_1, \dots, y_k), \\ T &\subseteq \{(f, g) \in \bar{V} \times \bar{V} : f(x_i) = g(x_i) \ \forall i = 1, \dots, k\}. \end{aligned}$$

Let's first note that if  $T$  has property P2 for  $k(T) > 0$ , then it also has it for  $k = 0$ . For practical reasons we want this  $k$  to increase, this is what allows for a better refinement. We mention this in hopes of avoiding any confusion. In aiming to define a sequence of coverings of  $D$ , we start by defining  $\mathcal{T}_0 := \{(\bar{V} \times \bar{V})_{V \times V}\}$ . Observe that  $D \subseteq V \times V \subseteq (\bar{V} \times \bar{V})_{V \times V}$  by lemma 5.2.1. The set  $T \in \mathcal{T}_0$  satisfies P1, P2 by letting  $k = 0, \Lambda = \emptyset$ . For some covering  $\mathcal{T}$  of  $D$  whose elements satisfy the refinement properties, we may define a refinement as follows for each each its elements.

**Definition 5.2.5** (The Refinement Procedure). *Let  $T$  be a set that has the covering property with  $k, \Lambda, x_i, y_i$  from P2 as in definition 5.2.4. Then we define its refinement  $\mathfrak{M}_T$  as follows,*

**Case 1:** *If  $T \subseteq \Delta$ , let  $\mathfrak{M}_T := \{T\}$ . Then  $T \in \mathfrak{M}_T$  has the covering property with the same  $k, \Lambda, x_i, y_i$ .*

**Case 2:** *If  $T \not\subseteq \Delta$ , for  $x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}), y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})$  let*

$$\begin{aligned} S_{xy} &:= T \cap (K(\mathbb{Z}_*^3 \setminus \Lambda, x_1, \dots, x_k, x) \times K(\mathbb{Z}_*^3 \setminus \Lambda, y_1, \dots, y_k, y)), \\ T_{xy} &:= [S_{xy} \cap \{(f, g) \in \bar{V} \times \bar{V} : f(x) = g(y)\}]_{V \times V}. \end{aligned}$$

We define for  $k, \Lambda, x_i, y_i$  belonging to  $T$ ,

$$\mathfrak{M}_T := \bigcup_{\substack{x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) \\ y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})}} \{T_{xy}\}.$$

Finally, each  $T_{xy}$  has the covering property with variables as follows. Let  $\Lambda_{xy} = \Lambda(T_{xy}) \in \{\emptyset, \langle e_1, e_2 \rangle_{\mathbb{Z}}, \langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}\}$  be maximal with  $f|_{\Lambda_{xy}} = g|_{\Lambda_{xy}}$  for all  $(f, g) \in T_{xy}$ . Let  $k(T_{xy}) = k(T) + 1$  and  $x_{k+1} = x, y_{k+1} = y$ . If  $\Lambda_{xy} = \Lambda$  we are done. In the case that  $\Lambda_{xy} \neq \Lambda$ , we instead let  $k(T_{xy})$  be the maximal number  $0 \leq r \leq k + 1$  with  $\#\(\{x_1, \dots, x_r\} \setminus \Lambda_{xy}\) = \#\(\{y_1, \dots, y_r\} \setminus \Lambda_{xy}\)$ , and  $x_1, \dots, x_r$  and  $y_1, \dots, y_r$  be the new sequences in  $\mathbb{Z}_*^3 \setminus \Lambda_{xy}$  belonging to  $T_{xy}$ .

This is the procedure that Schiemann stated in his summary, with the exception that he replaces  $S_{xy}$  with  $[S_{xy}]_{V \times V}$  when defining  $T_{xy}$ . We have found that it does not matter if we do that or not. The sets  $S_{xy}$  have the property that they decompose  $T$  into possibly overlapping subsets whose union is equal to  $T$ . When going from  $S_{xy}$  to  $T_{xy}$  we cut off redundant parts of the polyhedral cone. This is for example the sole purpose of the  $V \times V$  algorithm. Observe that each  $T_{xy}$  has the covering property, which follows directly by keeping lemma 5.2.1 and 5.2.2 in mind. With this refinement procedure, we are ready to define the sequence  $\mathcal{T}_i$  of coverings.

**Definition 5.2.6** (Refinement of a Covering &  $\mathcal{T}_i$ ). *If  $\mathcal{T}$  is a covering of  $D$ , then we define its refinement as*

$$\mathcal{T}' := \bigcup_{T \in \mathcal{T}} \mathfrak{M}_T.$$

With  $\mathcal{T}_0 := \{(\bar{V} \times \bar{V})_{V \times V}\}$ , we define the sequence  $\mathcal{T}_i$  by  $\mathcal{T}_{i+1} = \mathcal{T}'_i$  for each  $i \geq 0$ .

Here,  $\bar{V} \times \bar{V}$  has the covering property with  $\Lambda = \emptyset$  and  $k = 0$ . We must now argue that  $\mathcal{T}'$  is actually a refinement of a given covering  $\mathcal{T}$ . This is done via the following proposition.

**Proposition 5.2.7.** *The sequence  $\mathcal{T}_i$  is a sequence of coverings of  $D$  and each iteration is a refinement of the previous one.*

*Proof.* Noting that  $D \subseteq \overline{V} \times \overline{V}$ , we continue by induction. Assume that  $\mathcal{T} = \mathcal{T}_i$  for some  $i$  is a covering of  $D$ . We show that  $\mathcal{T}' = \mathcal{T}_{i+1}$  is also a covering of  $D$ . Fix some arbitrary  $T \in \mathcal{T}$ . We are done if we can show that

$$T \supseteq \bigcup_{T_{xy} \in \mathfrak{M}_T} T_{xy} \supseteq T \cap D,$$

since then

$$\bigcup_{T \in \mathcal{T}} T \supseteq \bigcup_{T \in \mathcal{T}} \left( \bigcup_{T_{xy} \in \mathfrak{M}_T} T_{xy} \right) \supseteq \bigcup_{T \in \mathcal{T}} (T \cap D) = D \cap \bigcup_{T \in \mathcal{T}} T = D,$$

by the fact that  $\mathcal{T}$  is a covering of  $D$ . Let us first note that by lemma 5.1.12,

$$\bigcup_{\substack{x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) \\ y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})}} K(\mathbb{Z}_*^3 \setminus \Lambda, x_1, \dots, x_k, x) \times K(\mathbb{Z}_*^3 \setminus \Lambda, y_1, \dots, y_k, y) = K(\mathbb{Z}_*^3 \setminus \Lambda, x_1, \dots, x_k) \times K(\mathbb{Z}_*^3 \setminus \Lambda, y_1, \dots, y_k),$$

implying that by property P2 of  $T$  as in definition 5.2.4 we have

$$\bigcup_{\substack{x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) \\ y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})}} S_{xy} = T \cap K(\mathbb{Z}_*^3 \setminus \Lambda, x_1, \dots, x_k) \times K(\mathbb{Z}_*^3 \setminus \Lambda, y_1, \dots, y_k) = T.$$

Consider  $(f, g) \in S_{xy} \cap D$ . We have  $\mathcal{R}(f, t) = \mathcal{R}(g, t)$  for all  $t \in \mathbb{R}_0^+$ . Since  $f|_\Lambda = g|_\Lambda$  we have  $\mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(f, t) = \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(g, t)$  for all  $t \in \mathbb{R}_0^+$ . By careful study of definition 5.1.10 and the definition of  $S_{xy}$ , we have for the corresponding values of  $x, y$

$$\begin{aligned} f(x) &= \min f(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) = \min \left\{ t_0 \in \mathbb{R}_0^+ : \sum_{0 \leq t \leq t_0} \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(f, t) \geq k + 1 \right\} \\ &= \min \left\{ t_0 \in \mathbb{R}_0^+ : \sum_{0 \leq t \leq t_0} \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda}(g, t) \geq k + 1 \right\} = \min g(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\}) = g(y), \end{aligned}$$

This implies that  $S_{xy} \cap D \subseteq T_{xy}$ , due to lemma 5.2.1, and the fact that  $D \subseteq V \times V$ . By what we did above, it follows that

$$T \supseteq \bigcup_{\substack{x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) \\ y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})}} T_{xy} \supseteq \bigcup_{\substack{x \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{x_1, \dots, x_k\}) \\ y \in \text{MIN}(\mathbb{Z}_*^3 \setminus \Lambda \setminus \{y_1, \dots, y_k\})}} (S_{xy} \cap D) = T \cap D. \quad \ominus$$

We devote the next definition and lemma to the following result, which serves as a motivation for why we should believe that  $\mathcal{T}_i$  becomes stationary:

$$\bigcap_{i \in \mathbb{N}} \bigcup_{T \in \mathcal{T}_i} T \cap (V \times V) = D.$$

**Definition 5.2.8.** Given  $f \in V$  and  $k \in \mathbb{N}$ , let

$$\psi(f, k) = \max \left\{ t \in f(\mathbb{Z}_*^3) \cup \{0\} : \sum_{s \leq t} \mathcal{R}_{\mathbb{Z}_*^3}(f, s) < k \right\}.$$

This function is well-defined since  $f$  positive definite and  $\psi(f, k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

**Lemma 5.2.9.** Let  $i \geq 3, T \in \mathcal{T}_i$  and  $(f, g) \in T \cap (V \times V)$ . We have

$$\mathcal{R}_{\mathbb{Z}_*^3}(f, t) = \mathcal{R}_{\mathbb{Z}_*^3}(g, t) \quad \forall t \leq \psi(f, \lfloor i/3 \rfloor - 1).$$

*Proof.* If  $T \subseteq \Delta$ , then we are already done. Assume instead that  $T \not\subseteq \Delta$ . Let  $(\overline{V} \times \overline{V})_{V \times V} = T_0 \supseteq \dots \supseteq T_i = T$  be a sequence of elements of coverings from  $\mathcal{T}_0, \dots, \mathcal{T}_i$ ; the sequence exists by construction. Let  $k_j = k(T_j)$ ,  $\Lambda_j = \Lambda(T_j)$  as in P2 for this sequence.  $\Lambda_j$  increases monotonely and takes at most 3 values. It follows that there

is a sequence  $\Lambda_{j_0} = \Lambda_{j_0+1} = \dots = \Lambda_{j_1}$  of length  $j_1 + 1 - j_0 \geq (i + 1)/3$  (since we have in total  $\Lambda_0, \dots, \Lambda_i$ ). The refinement procedure says that if  $T_j \not\subseteq \Delta$  and  $\Lambda_j = \Lambda_{j+1}$ , then  $k_{j+1} = k_j + 1$ , so that  $k_{j_1} = k_{j_0} + j_1 - j_0 \geq \lfloor i/3 \rfloor - 1$ . Let  $x_1, \dots, x_{k_{j_1}}, y_1, \dots, y_{k_{j_1}} \in \mathbb{Z}_*^3 \setminus \Lambda_{j_1}$  be corresponding values as in P2 for  $T_{j_1}$ . For all  $t < f(x_{k_{j_1}})$  we have using  $T_{j_1} \subseteq K(\mathbb{Z}_*^3 \setminus \Lambda_{j_1}, x_1, \dots, x_{k_{j_1}}) \times K(\mathbb{Z}_*^3 \setminus \Lambda_{j_1}, y_1, \dots, y_{k_{j_1}})$  that  $t < \min f(\mathbb{Z}_*^3 \setminus \Lambda_{j_1} \setminus \{x_1, \dots, x_{k_{j_1}}\})$ . For such a  $t$  we have  $\#\{1 \leq j < k_{j_1} : f(x_j) = t\} = \#\{x \in \mathbb{Z}_*^3 \setminus \Lambda_{j_1} : f(x) = t\}$ . Further, by P2,  $f(x_j) = g(y_j)$  for all  $1 \leq j \leq k(T_{j_1})$ . We get

$$\begin{aligned} \mathcal{R}_{\mathbb{Z}_*^3}(f, t) &= \mathcal{R}_{\mathbb{Z}_*^3 \cap \Lambda_{j_1}}(f, t) + \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda_{j_1}}(f, t) \\ &= \mathcal{R}_{\mathbb{Z}_*^3 \cap \Lambda_{j_1}}(f, t) + \#\{1 \leq j < k_{j_1} : f(x_j) = t\} \\ &= \mathcal{R}_{\mathbb{Z}_*^3 \cap \Lambda_{j_1}}(g, t) + \#\{1 \leq j < k_{j_1} : g(x_j) = t\} \\ &= \mathcal{R}_{\mathbb{Z}_*^3 \cap \Lambda_{j_1}}(g, t) + \mathcal{R}_{\mathbb{Z}_*^3 \setminus \Lambda_{j_1}}(g, t) \\ &= \mathcal{R}_{\mathbb{Z}_*^3}(g, t). \end{aligned}$$

Since  $\mathcal{R}_{\mathbb{Z}_*^3}(f, t) = \mathcal{R}_{\mathbb{Z}_*^3}(g, t)$  holds for all  $t < f(x_{k_j})$ , our statement follows from  $\psi(f, \lfloor i/3 \rfloor - 1) \leq \psi(f, k_{j_1})$  by the fact that  $\psi$  grows monotonely and that  $\psi(f, k_{j_1}) < f(x_{k_j})$ , and we now argue for the latter. Saying that  $\sum_{s \leq t} \mathcal{R}_{\mathbb{Z}_*^3}(f, s)$  should be less than  $k_{j_1}$  as in the definition of  $\psi(f, k_{j_1})$  means that  $t$  has to be less than  $f(x_{k_{j_1}})$ . This is because the sum counts the number of values and multiplicities of  $f$  over  $\mathbb{Z}_*^3$  and  $f(x_1) \leq \dots \leq f(x_{k_{j_1}})$ , where each  $x_i \in \mathbb{Z}_*^3$ .  $\circledast$

### 5.3 Part 3 – Schiemann’s Results

We summarize the results that Schiemann gives in this PhD thesis and summary. Some of the results will not be explained further in this report, since we are only really interested in proving theorem 5.1.

**Theorem 5.3.1** (Stability of  $\mathcal{T}_i$ ). *The sequence  $\mathcal{T}_i$  becomes stationary for  $i \geq 15$ . Further, we have that each set  $T \in \mathcal{T}_{15}$  lies in  $\Delta$ .*

Schiemann writes that the algorithm becomes stationary for  $i \geq 14$  in both his thesis and summary. The reason is that he chooses another  $\mathcal{T}_0$  in his thesis that skips the first iteration compared to ours. Nevertheless, this proves the third part of Schiemann’s theorem since  $\mathcal{T}_{15}$  is a covering of  $D$ . Details of the algorithm will be given in section 5.4 and 5.5, and the code we used is provided in appendix D. Schiemann also concerned himself with finding an explicit bound  $b(f)$  such that for any  $f, g \in V$  we have that

$$\mathcal{R}(f, t) = \mathcal{R}(g, t) \quad \forall t \leq b(f) \Rightarrow f = g.$$

For this is we introduce the following definition,

**Definition 5.3.2** (Successive Minima). *For a positive definite  $n$ -dimensional quadratic form  $f$  we define its successive minima for  $1 \leq i \leq n$  as*

$$s_i(f) = \min \left\{ f(x) : \exists x_1, \dots, x_i \in \mathbb{Z}^n \text{ that are linearly independent with } \max_j f(x_j) \leq f(x) \right\}.$$

For a Minkowski reduced form we have  $s_i(f) = f_{ii}$  as long as  $n \leq 4$  [16, p. 278]. The following two theorems are presented by Schiemann in his summary,

**Theorem 5.3.3** (Bound for Integral Equivalence). *Let  $f, g$  be ternary positive definite forms with real coefficients and let  $s_i = s_i(f)$  be the successive minima of  $f$ . Let*

$$b(f) = \min\{-1/14s_1 + 18/7s_2 + s_3, 3/2s_1 - 5/6s_2 + 17/6s_3, 13/5s_1 + s_2 + s_3, 7/2s_3\},$$

and  $\mathcal{R}(f, t) = \mathcal{R}(g, t)$  for  $t \leq b(f)$ , then  $f$  and  $g$  are integrally equivalent.

**Theorem 5.3.4** (Bound for Integral Equivalence when  $\det(f) = \det(g)$ ). *Let  $f, g$  be ternary positive definite forms with real coefficients and  $\det(f) = \det(g)$  and let  $s_i = s_i(f)$  be the successive minima of  $f$ . Let*

$$\begin{aligned} b(f) &= \min\{s_1 - s_2 + 3s_3, 11/13s_1 - 6/13s_2 + 34/13s_3, -s_1 + 2s_2 + 2s_3, \\ &\quad 4/3s_1 + 1/3s_2 + 5/3s_3, -2/3s_1 + 3s_2 + s_3, 14/9s_1 + s_2 + s_3, 3s_3\}, \end{aligned}$$

and  $\mathcal{R}(f, t) = \mathcal{R}(g, t)$  for  $t \leq b(f)$ , then  $f$  and  $g$  are integrally equivalent.

## 5.4 Part 4 – Computing $\text{MIN}(X)$

To determine  $\text{MIN}(X)$  for each  $X \subseteq \mathbb{Z}^3$  in our algorithm, we proceed as follows. We start by using lemma 5.4.1 *iii*) to give a faster way for the computer to calculate  $\text{MIN}(X)$  for finite sets  $X \subseteq \mathbb{Z}_*^3$ , even though it might not be necessary to use it. As long as  $X$  is finite, the calculation is nevertheless straightforward since we only have a finite amount of conditions. For infinite sets, we refer to corollary 5.4.5. We introduce some notation. For  $x \in \mathbb{Z}^3$  and for  $i \in \{1, 2, 3\}$ , we shall let  $d_i : \mathbb{Z}^3 \rightarrow \mathbb{Z}$  be given by

$$d_i(x) := \begin{cases} x_1x_2, & \text{if } i = 1, \\ x_1x_3, & \text{if } i = 2, \\ x_2x_3, & \text{if } i = 3. \end{cases}$$

**Lemma 5.4.1.** *For  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3) \in \mathbb{Z}^3$ , the following are equivalent*

(i)  $x \preceq y$ ,

(ii)  $y_3^2 - x_3^2 \geq 0$ ,  $y_3^2 - x_3^2 - |y_2y_3 - x_2x_3| + y_2^2 - x_2^2 \geq 0$  and for each  $(\delta_1, \delta_2, \delta_3) \in M'_3$ , the following holds

$$\|y\|^2 - \|x\|^2 + \sum_{i=1}^3 \delta_i(d_i(y) - d_i(x)) \geq 0,$$

(iii)  $y_3^2 - x_3^2 \geq 0$ ,  $y_3^2 - x_3^2 - |y_2y_3 - x_2x_3| + y_2^2 - x_2^2 \geq 0$  and

$$\begin{aligned} & \|y\|^2 - \|x\|^2 + \min(0, d_1(y) - d_1(x)) + \min(0, d_2(y) - d_2(x)) - |d_3(y) - d_3(x)| + \\ & + \max[0, \min(-(d_1(y) - d_1(x)), -(d_2(y) - d_2(x)), d_3(y) - d_3(x))] \geq 0. \end{aligned}$$

We recall the definition  $M'_3$  as the set  $\{(0, 0, \pm 1), (0, 1, \pm 1), (1, 0, \pm 1), (1, 1, 0), (1, 1, 1)\}$ .

*Proof.* Note that by proposition 5.1.9,  $x \preceq y$  if and only if  $f(x) \leq f(y)$  for all  $f \in M$ .

(i)  $\Leftrightarrow$  (ii) :

$\Rightarrow$ ) The three forms of  $M_1, M_2$  directly give the two first assumptions of (ii). The rest follows from insertion by elements of  $M_3$ . To see this, take

$$f = \begin{bmatrix} 2 & a & b \\ a & 2 & c \\ b & c & 2 \end{bmatrix}.$$

We have that  $f(x) \leq f(y)$  can be written

$$2\|x\|^2 + 2(ax_1x_2 + bx_1x_3 + cx_2x_3) \leq 2\|y\|^2 + 2(ay_1y_2 + by_1y_3 + cy_2y_3),$$

or equivalently

$$\|y\|^2 - \|x\|^2 + a(d_1(y) - d_1(x)) + b(d_2(y) - d_2(x)) + c(d_3(y) - d_3(x)) \geq 0.$$

We are done since the non-diagonal elements of elements of  $M_3$  perfectly align with elements of  $M'_3$ .

$\Leftarrow$ ) By the first direction, we showed that the conditions of (ii) are precisely those such that  $f(x) \leq f(y)$  for each  $f \in M$  which suffices.

(ii)  $\Leftrightarrow$  (iii) : Note that the first two conditions in (ii) and (iii) are equal. For any  $c_1, c_2, c_3 \in \mathbb{R}$ , we show that

$$\begin{aligned} m_1 & := \min \left\{ \sum_{i=1}^3 \delta_i c_i : (\delta_1, \delta_2, \delta_3) \in M'_3 \right\} \\ & = \min(0, c_1) + \min(0, c_2) - |c_3| + \max(0, \min(-c_1, -c_2, c_3)) =: m_2. \end{aligned}$$

As a direct consequence, (ii) and (iii) are equivalent. We split the proof of  $m_1 = m_2$  into two cases as follows.

**Case 1:**  $c_1, c_2 < 0$  and  $c_3 > 0$ .

Examining the elements of  $M'_3$  it is clear that the only of its elements that can give a minima in our case is one of  $(0, 1, -1), (1, 0, -1), (1, 1, 0)$ . The minima is  $c_2 - c_3, c_1 - c_3, c_1 + c_2$  respectively, which can be rewritten  $-|c_2| - |c_3|, -|c_1| - |c_3|, -|c_1| - |c_2|$ . In the first case  $|c_1|$  must have the smallest absolute value out of  $|c_1|, |c_2|, |c_3|$ , in the second case it must be  $|c_2|$  and in the third it is  $|c_3|$ . Therefore we have

$$m_1 = -|c_1| - |c_2| - |c_3| + \min(|c_1|, |c_2|, |c_3|) = c_1 + c_2 - c_3 + \min(|c_1|, |c_2|, |c_3|),$$

which is equal to  $m_2$  by definition.

**Case 2:**  $c_1 \geq 0$  or  $c_2 \geq 0$  or  $c_3 \leq 0$ .

In this case we have  $\max(0, \min(-c_1, -c_2, c_3)) = 0$ . Say  $c_1 \geq 0$ . It is not hard to see that only  $(0, 0, \pm 1), (0, 1, \pm 1)$  are possible minimas. If additionally  $c_2 > 0$ , then the minima is one of  $(0, 0, \pm 1)$ , and otherwise it is one of  $(0, 1, \pm 1)$ . This gives in both cases  $m_1 = \min(0, c_1) + \min(0, c_2) - |c_3|$ , which is equal to  $m_2$  by definition. Analogously, we have the same equality in the case that  $c_2 \geq 0$ . Finally, if  $c_3 \leq 0$ , we are left with the only possibilities  $(0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)$  for a minima. We have  $m_2 = \min(0, c_1) + \min(0, c_2) + c_3$ , from which  $m_1 = m_2$  follows.  $\odot$

Now that we have a faster way to calculate  $\text{MIN}(X)$  for finite sets  $X$ , we move on to the infinite case. We begin by proving two lemmas, after which we give the necessary information to calculate  $\text{MIN}(X)$  for relevant infinite sets  $X$ .

**Lemma 5.4.2.** *Let  $f \in M_3$  and  $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ . We have  $f(x) \geq \|x\|_\infty^2$ , where  $\|x\|_\infty = \max\{|x_i|\}$ .*

*Proof.* Since  $f \in M_3$ , we have  $f_{ii} = 2$  and  $f_{ij} \in \{-1, 0, 1\}$  for  $i \neq j$ . Let  $\{i, j, k\} = \{1, 2, 3\}$  where  $i$  denotes and index with  $\|x_i\| = \|x\|_\infty^2$ . It follows that  $|x_j x_k| = \min\{|x_1 x_2|, |x_1 x_3|, |x_2 x_3|\}$ , meaning that  $-|x_j x_k| \geq -|x_j x_i|$  and  $-|x_j x_k| \geq -|x_i x_k|$ . For  $f \in M_3$ , we have that either  $(f_{12}, f_{13}, f_{23}) = (1, 1, 1)$  or at least one of  $f_{12}, f_{13}, f_{23}$  is 0. In both cases at most two of the terms  $f_{12}x_1x_2, f_{13}x_1x_3, f_{23}x_2x_3$  can be negative and their sum has the lower bound  $-|x_j x_i| - |x_i x_k|$ . We deduce,

$$\begin{aligned} f(x) &= \sum_{n=1}^3 f_{nn}x_n^2 + 2 \sum_{n < m} f_{nm}x_nx_m \geq \left( \sum_{n=1}^3 2x_n^2 \right) - 2|x_j x_i| - 2|x_i x_k| = \\ &= 2(|x_i|/2 - |x_k|)^2 + 2(|x_i|/2 - |x_j|)^2 + x_i^2 \geq x_i^2 = \|x\|_\infty^2. \end{aligned} \quad \odot$$

**Lemma 5.4.3.** *Let  $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$  and  $a \in \mathbb{Z}$ .*

- a) *If  $x \not\prec (a, 1, 0)$ , then  $\|x\|_\infty < \sqrt{2(a^2 + \max(0, a) + 1)}$ ,*
- b) *If  $x_3 \neq 0$  and  $x \not\prec (a, 0, 1)$ , then  $\|x\|_\infty < \sqrt{2(a^2 + \max(0, a) + 1)}$ ,*
- c) *If  $x_2 \neq 0, x_3 \neq 0$  and  $x \not\prec (a, \text{sgn}(x_2x_3), 1)$ , then  $\|x\|_\infty < \sqrt{2(a^2 + |a| + \max(0, a) + 3)}$ .*

*Proof.*

a) If  $(x_2, x_3) = 0$ , then since  $x \in \mathbb{Z}_*^3$ ,  $x_1$  must be equal to 1, in which case we are done. So assume  $(x_2, x_3) \neq (0, 0)$ . Note that for  $f \in M_1 \cup M_2$ ,  $f(x) \geq f(a, 1, 0)$  as long as  $(x_2, x_3) \neq (0, 0)$ . This is clear for  $M_1$ , and for  $M_2$ , we have  $f(a, 1, 0) = 2$  for any  $a \in \mathbb{Z}$  and  $f(x) \neq 0$  since the lower left blockmatrices are positive definite, and its not hard to check that  $f(x)$  is an even positive integer. Therefore,  $x \not\prec (a, 1, 0)$  implies  $f(x) < f(a, 1, 0)$  for some  $f \in M_3$ . By definition, for any such  $f$ ,  $f(a, 1, 0) \in \{2a^2 + 2, 2a^2 + 2a + 2\}$ . By lemma 5.4.2,  $\|x\|_\infty^2 \leq f(x) < 2(a^2 + \max(0, a) + 1)$ .

b) This argument is analogous to that of a).

c) It is clear that for the element  $f$  of  $M_1$ ,  $f(x) \geq f(a, \text{sgn}(x_2x_3), 1)$ . Let now  $f$  be one of the forms in  $M_2$ . We have  $f(a, \text{sgn}(x_2x_3), 1) \in \{4 \pm 2 \text{sgn}(x_2x_3)\} = \{2, 6\}$ . As we noted in a),  $f(x)$  is an even positive integer since  $(x_2, x_3) \neq (0, 0)$ , so we only need to check when  $f(a, \text{sgn}(x_2x_3), 1) = 6$ . In this case,  $\pm \text{sgn}(x_2x_3) = 1$ . Now consider

$$f(x) = 2x_2^2 + 2x_3^3 \pm 2|x_2x_3| \text{sgn}(x_2x_3) = 2x_2^2 + 2x_3^3 + 2|x_2x_3| \geq 6,$$

since both  $x_2$  and  $x_3$  are non-zero. This implies  $f(x) < f(a, \text{sgn}(x_2x_3), 1)$  for some  $f \in M_3$ . By looking at the diagonals, we see that for such an  $f$ ,  $f(a, \text{sgn}(x_2x_3), 1) = 2a^2 + 4 + \lambda$  where  $\lambda$  is an element of

$$\{\pm 2 \text{sgn}(x_2x_3), 2(a \pm \text{sgn}(x_2x_3)), 2(a \text{sgn}(x_2x_3) \pm \text{sgn}(x_2x_3)), 2(a + a \text{sgn}(x_2x_3)), 2(a + a \text{sgn}(x_2x_3) + \text{sgn}(x_2x_3))\}.$$

We end by observing that this gives the upper bound  $2|a| + 2 \max(0, a) + 2$  for  $\lambda$ , meaning by lemma 5.4.2,  $\|x\|_\infty^2 \leq f(x) < 2a^2 + 2|a| + 2 \max(0, a) + 6$ .  $\odot$

Leading up to corollary 5.4.4 which we use whenever we calculate  $\text{MIN}(X)$  in our algorithm, we define the following three sets, all of which can be pre-computed and stored for values of  $a$  up to some number. According to Schiemann, it turns out that values  $|a| \leq 3$  are enough.

$$\begin{aligned} W_{0,a} &:= \{x \in \mathbb{Z}_*^3 : x \not\preceq (a, 1, 0)\} \cup \{(a, 1, 0)\}, \\ W_{1,a} &:= \{x \in \mathbb{Z}_*^3 \setminus \langle e_1, e_2 \rangle_{\mathbb{Z}} : x \not\preceq (a, 0, 1)\} \cup \{(a, 0, 1)\}, \\ W_{2,a} &:= \{x \in \mathbb{Z}_*^3 \setminus \langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}} : x \not\preceq (a, \pm 1, 1)\} \cup \{(a, \pm 1, 1)\}. \end{aligned}$$

By Lemma 5.4.3, we get a good idea of how to compute each  $W_{i,a}$  by obtaining the following inclusions in finite sets.

$$\begin{aligned} W_{0,a} &\subseteq \{x : \|x\|_\infty < \sqrt{2(a^2 + \max(0, a) + 1)}\}, \\ W_{1,a} &\subseteq \{x : \|x\|_\infty < \sqrt{2(a^2 + \max(0, a) + 1)}\}, \\ W_{2,a} &\subseteq \{x : \|x\|_\infty < \sqrt{2(a^2 + |a| + \max(0, a) + 3)}\}. \end{aligned}$$

**Lemma 5.4.4.** *Let  $\emptyset \neq Y \subseteq X \subseteq \mathbb{Z}_*^3$ . If  $W \supseteq \{x \in X : x \not\preceq y \ \forall y \in Y\} \cup Y$ , then  $\text{MIN}(X) = \text{MIN}(X \cap W)$ .*

*Proof.* We first show  $\text{MIN}(X) \subseteq \text{MIN}(X \cap W)$ . Take  $x_0 \in \text{MIN}(X)$ . If  $x_0 \not\preceq y$  for all  $y \in Y$ , then by definition of  $W$ ,  $x_0 \in W$ . If for some  $y_0 \in Y$  we have  $x_0 \succeq y_0$  then by property of  $\text{MIN}(X)$  we have  $y = x \in Y$ , meaning that in any case we have  $x_0 \in W$ . Since  $x_0 \in \text{MIN}(X)$  we have by definition that for each  $x \neq z \in X$ ,  $z \not\preceq x$ . It follows each  $x \neq z \in X \cap W \subseteq X$  also has this property. By definition,  $x_0 \in \text{MIN}(X \cap W)$ .

Secondly we show  $\text{MIN}(X) \supseteq \text{MIN}(X \cap W)$ . Let  $x_0 \in \text{MIN}(X \cap W)$  and take any  $x \in X$  such that  $x \preceq x_0$ . If  $x$  is necessarily equal to  $x_0$ , then  $x_0 \in \text{MIN}(X)$  by definition. If  $x \in X \cap W$ , then since  $x_0 \in \text{MIN}(X \cap W)$ , we have  $x_0 = x$ . If  $x \in X \setminus W$ , then it follows that there is a  $y \in Y$  such that  $y \preceq x$ . By transitivity,  $y \preceq x_0$ . Now since  $Y \subseteq X \cap W$ , this implies  $y = x_0$ . As a consequence,  $x_0 \preceq x \preceq x_0$  which by proposition 5.1.9 means  $x_0 = -x$ , but since we are in  $\mathbb{Z}_*^3$  we must have  $x_0 = x$ .  $\odot$

**Corollary 5.4.5.** *Let  $X \subseteq \mathbb{Z}_*^3$  and  $a$  be any integer. Then we have*

- a)  $(a, 1, 0) \in X \Rightarrow \text{MIN}(X) = \text{MIN}(X \cap W_{0,a})$ ,
- b)  $X \cap \langle e_1, e_2 \rangle_{\mathbb{Z}} = \emptyset$  and  $(a, 0, 1) \in X \Rightarrow \text{MIN}(X) = \text{MIN}(X \cap W_{1,a})$ ,
- c)  $X \cap (\langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}) = \emptyset$  and  $(a, \pm 1, 1) \in X \Rightarrow \text{MIN}(X) = \text{MIN}(X \cap W_{2,a})$ .

*Proof.* We get the statement by directly applying lemma 5.4.3 in the following way: a) Let  $Y = (a, 1, 0)$  and  $W = W_{0,a}$ , b) Let  $Y = (a, 0, 1)$  and  $W = W_{1,a}$ , c) Let  $Y = \{(a, -1, 1), (a, 1, 1)\}$  and  $W = W_{2,a}$ .  $\odot$

## 5.5 Part 5 – Algorithmic Considerations

Writing a computer program that proves Schiemann's theorem using his algorithms is quite a big project. Especially if we consider that there are many ways to alter the algorithm slightly to possibly make it run faster or to simplify it. We would like to share our experience with writing this program in this section and discuss some things that are worth keeping in mind, and other observations. First of all, we note that Schiemann did his computations in-depth first while we do a breadth first search. This should not matter in terms of time as long as one uses only one processor. The full code for our program is provided in appendix D. In this section, we give an overview of how we wrote the program. Firstly, we make use of the following datastructures where  $T = P_c(A, B)$  satisfies the covering property.

**Polyhedral Cone:** Consists of the two matrices  $A$  and  $B$ .

**Polyhedral Cone Info:** Consists of a polyhedral cone  $T$  and its most important information, namely its dimension and its edges.

**Polyhedral Cover:** Consists of the polyhedral cone info corresponding to  $T$ ,  $\Lambda(T)$  as an integer,  $k(T)$  as an integer, and the values of  $x_i, y_i$  as arrays of vectors from P2.

The covering  $\mathcal{T}$  will then be an array of elements of the polyhedral cover type. Admittedly, we don't need to store  $k$ , we only want to make sure that there are the same number of  $x_i$  as  $y_i$  corresponding to  $T$ . Taking product of polyhedral cones is done as in lemma 4.2.5, and intersection is done as in lemma 4.2.8, but where we use the algorithm of section 4.4, to get the new edges and dimension. However, we only ever need to add one row at a time to  $A$  and in application we don't intersect polyhedral cones like that. There are two parts of the algorithm that are non-trivial. The first is to calculate  $\text{MIN}(X)$  and the second is to check whether a polyhedral cone  $P_c(A, B)$  is included in a set of the form  $(a, b)^\perp$  and if it is included in  $\Delta$ . The first part was done in the previous section and the latter is explained in section 4.4. We realize the set  $\Delta$  of all identical pairs in  $\bar{V} \times \bar{V}$  as the set  $\Delta = P_c(A_\Delta, \emptyset)$  where  $A_\Delta$  is the set of vectors of the forms  $e_i - e_{6+i}$  and  $-e_i + e_{6+i}$  for  $1 \leq i \leq 6$ . We also embed pairs  $(f, g) \in T$  in  $\mathbb{R}^{12}$  by

$$(f, g) \mapsto (f_{11}, f_{22}, f_{33}, f_{12}, f_{13}, f_{23}, g_{11}, g_{22}, g_{33}, g_{12}, g_{13}, g_{23}).$$

If the edges of  $T$ , say  $k_i$ , all satisfy  $a \cdot k_i \geq 0$  for each  $a \in A_\Delta$ , then  $T \subseteq \Delta$ . To see this, note that for  $\lambda_i \geq 0$  we have  $a \cdot (\sum \lambda_i k_i) \geq 0$  and also note that any  $x \in T$  can be written  $\sum \lambda_i k_i$  for some  $\lambda_i \geq 0$ . To check if  $T \subseteq v^{\geq 0}, v^\perp$  respectively, we similarly only need to check that each  $k_i \in v^{\geq 0}, v^\perp$  respectively. We have the corresponding procedure to check whether  $T \subseteq \Lambda_1$  and  $T \subseteq \Lambda_2$ , where

$$\begin{aligned} \Lambda_1 &:= \{(f, g) \in \bar{V} \times \bar{V} : f|_{\langle e_1, e_2 \rangle_{\mathbb{Z}}} = g|_{\langle e_1, e_2 \rangle_{\mathbb{Z}}}\}, \\ \Lambda_2 &:= \{(f, g) \in \bar{V} \times \bar{V} : f|_{\langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}} = g|_{\langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}}\}. \end{aligned}$$

These sets can equivalently be written as follows,

$$\begin{aligned} \Lambda_1 &:= \{(f, g) \in \bar{V} \times \bar{V} : (f_{11}, f_{22}, f_{12}) = (g_{11}, g_{22}, g_{12})\}, \\ \Lambda_2 &:= \{(f, g) \in \bar{V} \times \bar{V} : (f_{11}, f_{22}, f_{33}, f_{12}, f_{13}) = (g_{11}, g_{22}, g_{33}, g_{12}, g_{13})\}. \end{aligned}$$

The sets  $\Lambda_1, \Lambda_2$  are defined as polyhedral cones in the same way as  $\Delta$  was. With this in mind, the procedure is straight-forward. We would however like to point out the importance of using the algorithm of section 4.4 to update the polyhedral cones. To begin with, we used polymake's own functions to determine whether a polyhedral cone lies inside another. Of course, these are well optimized functions that probably are as fast as they possibly can be. However, if we don't save the edges and the dimension of the polyhedral cones, then we have to start over with each calculation. When running the program we saw that as we used the algorithm from section 4.4, even as the matrices and arrays that define the polyhedral cone got longer, the time it took to get to the next iteration seemed to increase linearly with respect to the number of sets in the polyhedral covering. When we used only polymake's functions, we were not close to be able to finish the program in time. In comparison, the calculation of  $\text{MIN}(X)$  is negligible, meaning it is very fast, so perhaps we could completely skip lemma 5.4.1. We now present how long each iteration took and how many polyhedral cones not included in  $\Delta$  that were still in the sequence of polyhedral covers.

State after the $i$ :th iteration								
$i$	0	1	2	3	4	5	6	7
Number of $T \in \mathcal{T}_i : T \not\subseteq \Delta$	1	1	1	4	42	500	3,311	11,164
Time from start (in HH:MM)	00:00	00:00	00:00	00:00	00:00	00:02	00:05	00:13

State after the $i$ :th iteration								
$i$	8	9	10	11	12	13	14	15
Number of $T \in \mathcal{T}_i : T \not\subseteq \Delta$	31,334	59,970	34,658	4,452	1,284	702	18	0
Time from start (in HH:MM)	00:28	00:59	01:48	02:22	02:42	02:53	03:00	03:01



In other words, it took about 3 hours and we computed 147,442 polyhedral cones. We move on to discuss some interesting details of the program. For example, even though case 2.1 could theoretically be completely omitted and we would still get a valid sequence of coverings, it is very necessary to reduce the number of cases that we must calculate. When we tried to omit case 2.1, we got an unreasonable amount of polyhedral cones at stage  $i = 9$  and the program was not able to finish, even with multiple processors. We believe that there are two reasons. Firstly, by experience it seems to be generally the case that the number of sets in  $\text{MIN}(X \setminus \Lambda)$  is less than or equal (in terms of number of elements) to  $\text{MIN}(X)$  where  $\Lambda$  is one of the three choices as from the covering property. This would give us less iterations at that stage of the algorithm. Secondly, recall that in the definition of  $T_{xy}$ , we make an intersection with the set of pairs with  $f(x) = g(y)$ . Now if we already knew that  $f(x) = g(y)$ , then we would get less information from this step, and so we would need to do more iterations. When doing case 2.1, we greatly reduce the risk of that ever happening. As we mentioned, testing shows that case 2.1 is very important so one might wonder if it can be expanded upon. This is discussed in section 6.2.

To make the program even faster, we refer to Schiemann's thesis where he gives a more complicated algorithm that uses symmetrical properties of certain polyhedral cones. Using those algorithm, he computes about 120,000 polyhedral cones [3, p. 517] instead of almost 150,000 as we do. Another idea is to try to determine some properties of the pairs of  $D$  such that we can start with a smaller set than the most general  $\bar{V} \times \bar{V}$ . For example, using elementary methods such as in section 3.1, we can determined that  $(f, g) \in D$  implies  $f_{11} = g_{11}$  and  $f_{22} = g_{22}$ . Schiemann knew that  $f_{11} = g_{11}$  and therefore he let  $\mathcal{T}'_0 = \{[(\bar{V} \times \bar{V}) \cap (e_1 - e_7)^\perp]_{V \times V}\}$  (observe  $(e_1 - e_7)^\perp = \{x \in \mathbb{R}^{12} : x_1 = x_7\}$ ). Interestingly,  $\mathcal{T}'_0 = \mathcal{T}_1$  where  $\mathcal{T}_1$  denotes our covering after one iteration (this is why we get  $i \geq 15$  instead of  $i \geq 14$  in theorem 5.3.1). We can see in the table that it makes no difference in time either which way. We also believe that assuming  $f_{22} = g_{22}$  in our first covering would not greatly impact the total runtime.

We end with a discussion about the testing of our functions in the program. The algorithm that is given in section 4.4 can be tested by comparing it with polymake's function. Using this method, we have checked that our algorithm works for a small number of different polyhedral cones. Unfortunately, due to technical difficulties we were not able to do as much testing as we would have liked. Regarding the calculation of  $\text{MIN}(X)$ , we do not know of any available function to test this algorithm. Of courses we have carefully checked through all the code, but we have also checked it with a number of sets that can be calculated by hand. Also due to the fact that we got the equivalent number of iterations as Schiemann did, we are certain that our program works as intended.

# Chapter 6

## Going Forward

We have explained how the relation between the three different perspectives, the analytical properties of flat tori, the geometry of lattices and positive definite forms and their relation to Schiemann's theorem. We have also presented important information on this topic and we have proven a number of results, including Schiemann's theorem III which we hopefully made more accesible. Before we reach the end of this thesis, we discuss what can be done in the future.

### 6.1 Open Problems

There are a number of questions that we are convinced have still not been answered regarding the spectral geometry of flat tori, even though a lot of brilliant mathematicians have contributed to this field. Specifically, we do not know what the number  $N_n$  is when  $n \geq 4$ . We only know that it grows as in proposition 2.4.4. It is reasonable to think that using a similar method as in the proof for part III of Schiemann's theorem, we could show that  $N_4$  is a finite number. This will be discussed in section 6.2, and the first thing we want to do is embed  $q \in \delta_4^+$  in  $\mathbb{R}^{10}$ . Any which way is fine, but one way of doing is the following,

$$q \mapsto (q_{11}, q_{22}, q_{33}, q_{44}, q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34}).$$

We previously mentioned that the 7-dimensional case is of special importance. By lemma 2.4.4, we know that  $N_7 \geq 2$  and if we could show that  $N_7 = 2$ , then  $N_4 = N_5 = N_6 = 2$  would follow by Schiemann's lemma and we could conjecture that actually  $N_n = \lfloor n/4 \rfloor + 1$  for all  $n$ .

The conjecture we gave in section 3.3 is also unanswered to our knowledge. If true, it would say something about what pairs of lattices can be seemlesly *folded* into distinct higher dimensional ones. We don't currently have any ideas of how to attack that problem.

In 2011, Jahan Claes wrote a paper on the spectral determination of flat tori. Reading it, it is apparent that he was not aware of the third part of Schiemann's theorem, even though it certainly would have been relevant in the context of this report. This somewhat speaks to the obscurity of Schiemann's theorem. In any case, Claes introduced the following definition and proved the theorem of section 6.3 [23].

**Definition 6.1.1** (*k*-Spectrum). *For an  $n$ -dimensional lattice  $L$ , we define the  $k$ -spectrum of  $L$  to be*

$$\mathcal{L}^k(L) := \{(\Lambda, m_\Lambda) : \Lambda \text{ is a } k\text{-dimensional sub-lattice of } L\}.$$

*Here,  $m_\Lambda$  is the number of such sub-lattices of  $L$  that are congruent to  $\Lambda$ .*

By a  $k$ -dimensional sub-lattice we now refer to the general formulation that was given in section 1.4, meaning that it is euqual to  $\langle v_1, \dots, v_k \rangle_{\mathbb{Z}}$  for some lienarly independent vectors  $v_i \in L$ . It is not a coincidence that we use a similar notation as for the length spectrum;  $\mathcal{L}^1(L)$  can easily be identified as the length spectrum of  $L$ . The  $k$ -spectrum is therefore a generalization of it. We'll give a more in-depth discuss in section 6.4, and show how this definition leads to many new problems.

## 6.2 $N_4 \not\leq \infty$

In an attempt to prove that  $N_4 < \infty$ , we might consider using the techniques of chapter 5 and generalizing them. There are however a number of things we must consider. Let  $V' \subseteq \mathcal{M}_4$  be a set containing a representation of each 4-dimensional positive definite quadartic form. To show that  $N_4 < \infty$  we believe that we don't need all the representatives of  $V'$  to be unique in  $V'$ . This will instead be important in section 6.3. However, it is only fitting to note that the set  $\mathcal{M}_n$  is *almost* unique, by a theorem stated in [22] for which we give a short proof,

**Theorem 6.2.1** (Almost-Uniqueness of  $\mathcal{M}_n$ ). *For any  $q \in \mathcal{M}_n$  such that all corresponding, by theorem 4.3.4, non-redundant inequalities defining  $\mathcal{M}_n$  are strict,  $q$  has a unique representative in  $\mathcal{M}_n$  up to representatives  $q \circ B$  for elements  $B = \text{diag}(\chi_i) \in GL_n(\mathbb{Z})$  where  $\chi_i \in \{-1, 1\}$ .*

*Sketch of Proof.* It follows by assumption that all including the redundant inequalities defining  $\mathcal{M}_n$  are strict with respect to  $q$ , except those who are never strict, those one the form  $q(e_s) \geq q_{ss}$  for some  $s$ . Let  $B \in GL_n(\mathbb{Z})$  be such that  $q \circ B \in \mathcal{M}_n$ . Let  $k \geq 0$  be the biggest integer such that we can write

$$B = \begin{bmatrix} I_{k-1} & 0 \\ 0 & B_0 \end{bmatrix}$$

where  $B_0$  is a square matrix of dimension  $k$  and  $I_{k-1}$  is the identity matrix of dimension  $k-1$ . Consider the strict inequality  $q(x) > q_{kk}$  for some  $x \neq \pm e_k$  with  $\text{GCD}(x_k, \dots, x_n) = 1$ . Since  $q(x) > q(e_k)$  due to the strict inequality,  $x$  cannot be the  $k$ :th vector of  $B$ . This implies that  $x = \pm e_k$  and as a consequence,  $B = \text{diag}(\chi_i)$  as we wanted. To check that this  $B$  actually works, we refer to the fact that the strict inequality  $q(x) > q_{kk}$  should by assumption hold no matter how we change the sign of the coordinates of  $x$ .  $\odot$

The dimension of  $\mathcal{M}_n$  is equal to  $n(n+1)/2$ , and this theorem says that only forms that lie on some facet of  $\mathcal{M}_n$  can (up to base change of  $\text{diag}(\chi_i)$ ) have more than one representative in  $\mathcal{M}_n$ , which “qualitatively” speaking is negligible. By calculating the edges of  $V'$ , for example with a computer, we can in a natural way extend the concept of  $\text{MIN}(X)$  for  $X \subseteq \mathbb{Z}_*^n$  for any dimension  $n$ . Here,  $\mathbb{Z}_*^n$  is defined analogously as for  $\mathbb{Z}_*^3$ .

Since  $N_4 \geq 2$ , we must have a different termination criterion than that in chapter 5, since we cannot possibly have  $T \subseteq \Delta'$  for each  $T$  in any covering, where  $\Delta'$  in this case is the pairs of identical forms in  $V' \times V'$ . Instead, it would be enough to see that  $T \subseteq \nabla$ , where  $\nabla$  is the set of pairs  $(f, g)$  in  $V' \times V'$  with the property that the  $i$ :th coordinate of  $f$  uniquely determines the  $i$ :th coordinate in  $g$  and vice versa. Without going into the details, we mention that this is a reasonable termination criterion in any dimension due to theorem 3.4.5, that given some representation numbers, only a finite number of forms in  $\delta_n^+$  shares it up to equivalence. The mentioned termination criterion is easy to check, if we let  $k^1, \dots, k^r$  be the edges of  $T$ , then if the  $2 \times r$  matrix

$$\begin{bmatrix} k_i^1 & k_i^2 & \dots & k_i^r \\ k_{10+i}^1 & k_{10+i}^2 & \dots & k_{10+i}^r \end{bmatrix}$$

is of rank 1 (where  $k_i^j$  is the  $i$ :th coordinate of  $k^j$ ), we know that the  $i$ :th coordinate of the first form of a pair  $(f, g) \in T$  determines that of the second and vice versa. Just as in chapter 5, we should do this in stages, meaning that we should update our corresponding sets  $x_1, \dots, x_k, y_1, \dots, y_k$  as in case 2.1. We believe that if this is done optimally, the program will be fast enough. With knowledge of the computations of the 3-dimensional case in mind, we suggest that instead of letting  $\Lambda \in \{\emptyset, \langle e_1, e_2 \rangle_{\mathbb{Z}}, \langle e_1, e_2 \rangle_{\mathbb{Z}} \cup \langle e_1, e_3 \rangle_{\mathbb{Z}}\}$  be maximal with  $f|_{\Lambda} = g|_{\Lambda}$  for all  $(f, g) \in T$ , we consider  $\Lambda$  to be the maximal number  $0 \leq i \leq 10$  such that each coordinate up until the  $i$ :th of  $f$  determines that coordinate of  $g$  and vice versa, where  $(f, g) \in T$ . We believe that most of the statements and concepts in Schiemann's proof can without issue extended to 4 dimensions. With the consideration that we have provided in this section, it should be very possible to check whether  $N_4$  is finite. It is however a possibility that the algorithm would not stop, either because it is too slow or because  $N_4 = \infty$ , in which case we could still describe the sequence of quadratic forms that imply  $N_4 = \infty$ . If it does succeed, then we can also find an upper bound for  $N_4$ , using the same algorithm.

## 6.3 $N_4 \not\geq 2$

Let's say that we have done the above algorithm to find that  $N_4 < \infty$ . If we want to try to show that  $N_4 = 2$ , then we should use a set  $V \subseteq \mathcal{M}_n$  of unique representations all of positive definite forms. For this, we refer to

the lexicographic order explained in [24, p. 192]. We assume that  $V$  is such a set, further we may define,

**Definition 6.3.1** (Sign Reduced  $n$ -Dimensional Forms). *We let  $V_n \subseteq \mathcal{M}_n$  denote some set that includes a unique representation of all  $n$ -dimensional positive definite quadratic forms.*

We also make a general definition for the *representative set*. This will help us to make adjust Schiemann's algorithm in a satisfactory way.

**Definition 6.3.2** (The Representative Set). *Let  $k, n$  be positive integers. We then define*

$$D_n^k = \{(f_1, \dots, f_k) \in V_n : \mathcal{R}(f_1, t) = \dots = \mathcal{R}(f_k, t) \ \forall t \geq 0\}.$$

In our specific case, we will consider  $D_4^3$ , and we want to cover it with a triplet  $(f, g, h) \in V_4 \times V_4 \times V_4$ . We could without problems modify Schiemann's algorithm to deal with triplets instead. Our termination criterion in this case is that for  $T$ , each triplet  $(f, g, h)$  should have either  $f = g, f = h$  or  $g = h$ , since then there are no triplets of unique forms that all share representation numbers. To do this in stages, do the analogous steps as in section 6.2, but for triplets. However, if we work with triplets, then we will in each iteration go through three MIN sets instead of two. For this reason, the algorithm might be too slow. We could also try with pairs and  $D_4^2$  as in chapter 5 and proceed with the termination criterion as in section 6.2. However, then we must save those  $T$  that have terminated but are not in  $\Delta$  and check later if  $N_4 = 2$ . This would be done by going through all such  $T$  pair by pair. Let  $T_1, T_2$  be two such sets. For any two pairs  $(f_1, g_1) \in T_1, (f_2, g_2) \in T_2$  we want to check if  $f_1 = f_2$  and then if  $g_1 = g_2$ . More precisely, if there is a triplet  $(f, g, h)$  of distinct forms that make up two distinct pairs that are in  $T_1, T_2$  respectively, then we would discover this by looking at  $\text{Pr}_1(T_1) \cap \text{Pr}_1(T_2)$  and  $\text{Pr}_2(T_1) \cap \text{Pr}_2(T_2)$ , where  $\text{Pr}_1$  is the projection onto the first 10 coordinates and  $\text{Pr}_2$  is the projection onto the last 10. We describe the polyhedral cone  $P_c$  such that  $P_c = \text{Pr}_1(T_1) \cap \text{Pr}_1(T_2)$ , and then consider  $K_1 := (P_c \times \mathbb{R}^{10}) \cap T_1, K_2 := (P_c \times \mathbb{R}^{10}) \cap T_2$ . The final step is to look at the edges of  $K_1, K_2$  and checking whether they are equal. If there are edges  $k_1$  of  $K_1$  and  $k_2$  of  $K_2$  such that their  $\text{Pr}_1$ -projection are parallel, then we check if their  $\text{Pr}_2$ -projection also are parallel. If they are, then  $k_1 = k_2$  as edges, and otherwise we check if  $k_1 = (f, g)$  has  $f = g$  or  $k_2 = (f', g')$  has  $f' = g'$  (here  $f, f'$  are parallel and  $g, g'$  are not). If neither is true,  $N_4 \geq 3$ . Even though we just gave a sketch of the procedure, we believe this can be done effectively to determine whether  $N_4 = 2$ .

## 6.4 $k$ -Spectra

The  $k$ -spectrum is a generalization of the length spectrum and it gives rise to many new problems about lattice geometry. The most natural one is the following,

- Does the  $k$ -spectrum determine the shape of  $n$ -dimensional lattices?

We should think geometrically about this question as whether the parallelograms or parallelepipeds of a lattice determine its shape, in the same sense as whether its vectors do. The answer to the question is of course generally no. We have seen that the 1-spectrum only does so when  $n \leq 3$ . But what about the 2-spectrum? And what about the  $n - 1$ -spectrum?

To begin with we might ask whether the 2-spectrum determines 4-dimensional lattices. The obvious first step here would be to consider Conway's and Sloane's family of 4-dimensional isospectral non-congruent lattices. If we could prove that one of the pairs share their 2-spectrum, then the 2-spectrum would not determine the shape. And if none of the pairs in the family share it, then we could conjecture that it does determine the shape. Another natural question is if the  $n - 1$ -spectrum determines the shape of  $n$ -dimensional lattices. One argument that the answer should be yes, is that the  $n - 1$ -spectrum determines the facets of the parallelepiped that defines the lattice, which should be stronger than just knowing the lengths of it. Finally we ask for which pairs  $(a, b)$  does the  $a$ -spectrum determine the  $b$ -spectrum? We end by showing Claes's result, the proof of which is non-trivial but far simpler than that of the third part of Schiemann's theorem.

**Theorem 6.4.1** (Claes's Theorem). *If  $L_1, L_2$  are 3-dimensional lattices and  $\mathcal{L}^2(L_1) = \mathcal{L}^2(L_2)$ , then  $L_1 \stackrel{\mathcal{C}}{\sim} L_2$ .*

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# Appendix A

## Lattice Theory

**Lemma A.0.1.** *The set  $O_n(\mathbb{R})$  consists of matrices whose vectors are orthonormal bases of  $\mathbb{R}^n$ . Further, let  $A$  be linear transformation from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Then  $A \in O_n(\mathbb{R})$  if and only if  $A$  takes an orthonormal basis to another.*

*Proof.* Let  $C = [c_i] \in O_n(\mathbb{R})$ . We have  $C^T C = I$ , in other words  $C \in O_n(\mathbb{R})$  if and only if  $c_i^T c_j = \delta_{ij}$ , but this is the definition of an orthonormal basis  $\{c_1, \dots, c_n\}$  of  $\mathbb{R}^n$ . For the second part we consider two arbitrary orthonormal bases  $\{u_1, \dots, u_n\}$  and  $\{v_1, \dots, v_n\}$ . Let also  $U = [u_i]$  and  $V = [v_i]$ .

$\Rightarrow$ : Assume  $A \in O_n(\mathbb{R})$ . We are done if we can show  $AU \in O_n(\mathbb{R})$  and we have  $(AU)^T AU = U^T A^T AU = U^T U = I$ .

$\Leftarrow$ : By assumption we may assume  $AU = V$ . We check that  $VU^{-1} \in O_n(\mathbb{R})$  as follows,

$$(VU^{-1})^T VU^{-1} = U^{-T} V^T V U^{-1} = U^{-T} U^{-1} = (UU^T)^{-1} = I. \quad \odot$$

**Proposition A.0.2.** *To any  $A \in GL_n(\mathbb{R})$ , there is a  $C \in O_n(\mathbb{R})$  such that  $CA$  is an upper triangular matrix with positive diagonal elements.*

*Proof.* We do the proof by induction. If  $n = 1$ , then the proposition is trivial. Assume for some  $n \geq 1$  that the statement holds. Take an invertible matrix  $A \in \mathbb{R}^{(n+1) \times (n+1)}$ . The first  $n$  vectors span an  $n$ -dimensional subspace with a unit normal vector, say  $N$ . There is a  $C \in O_{n+1}(\mathbb{R})$  that rotates  $N$  to  $e_{n+1}$ . It follows by induction that there is also a  $C' \in O_{n+1}(\mathbb{R})$  such that

$$CA = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1,n} & a'_{1,n+1} \\ a'_{21} & a'_{22} & \cdots & a'_{2,n} & a'_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a'_{n,1} & a'_{n,2} & \cdots & a'_{n,n} & a'_{n,n+1} \\ 0 & 0 & \cdots & 0 & a'_{n+1,n+1} \end{bmatrix} \quad \& \quad C'CA = \begin{bmatrix} a''_{11} & a''_{12} & \cdots & a''_{1,n} & a''_{1,n+1} \\ 0 & a''_{22} & \cdots & a''_{2,n} & a''_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a''_{n,n} & a''_{n,n+1} \\ 0 & 0 & \cdots & 0 & a''_{n+1,n+1} \end{bmatrix},$$

Finally, to make sure that the diagonal elements are positive, we only need to left multiply  $C'CA$  with a diagonal orthonormal matrix that has either  $-1, 1$  as its diagonal entries depending on the sign of the diagonal entries of  $C'CA$ .  $\odot$

**Lemma A.0.3.** *The dual lattice is the set of linear bounded functions from  $\Gamma$  to  $\mathbb{Z}$ .*

*Proof.* We first observe that viewing the elements of the dual lattice as functions in the natural way, they are bounded and linear. Let now  $\Gamma = AZ^n$  and take a linear bounded function  $f : \Gamma \rightarrow \mathbb{Z}$ . Since  $f$  is linear and defined on the vectors of the invertible matrix  $A$ , we can uniquely extend it to a linear function  $\tilde{f}$  from  $\mathbb{R}^n$  to  $\mathbb{R}$ , viewing both sets as Hilbert spaces with the Euclidean inner product. Riesz representation theorem now says that  $\tilde{f}$  is represented by some vector, call it  $\gamma^*$ , in  $\mathbb{R}^n$  such that  $\tilde{f}(x) = \gamma^* \cdot x$  for  $x \in \mathbb{R}^n$ . Further,  $\tilde{f}|_{\Gamma} = f$  implying  $f(\gamma) = \gamma^* \cdot \gamma$  for all  $\gamma \in \Gamma$ .  $\odot$

**Definition A.0.4** (Sublattice). *Let  $\Gamma$  be lattice. A lattice  $\Lambda$  is a sublattice of  $\Gamma$  if  $\Lambda \subseteq \Gamma$ .*

**Proposition A.0.5.** Let  $\Lambda$  and  $\Gamma$  be lattices and let  $A_\Lambda, A_\Gamma$  be corresponding basis matrices.

i)  $\Lambda$  is a sublattice of  $\Gamma$  if and only if  $A_\Lambda = A_\Gamma V$  for some  $V \in \mathbb{Z}^{n \times n}$ .

ii) If  $\Lambda$  is a sublattice of  $\Gamma$ , then

$$\frac{\det(A_\Lambda)}{\det(A_\Gamma)} \Gamma,$$

is a sublattice of  $\Lambda$  and  $\det(A_\Lambda)/\det(A_\Gamma) \in \mathbb{Z} \setminus \{0\}$ .

iii) Let  $\Lambda$  be a sublattice of  $\Gamma$ . If  $\gamma \in \Gamma$ , then  $\Lambda \cap \text{Span}_{\mathbb{Z}}\{\gamma\} \neq 0$ .

*Proof.*

i)

$\Rightarrow$ ) Since  $\Lambda \subseteq \Gamma$ , each point of  $\Lambda$  lies in  $A_\Gamma \mathbb{Z}^n$ . In other words, any vectors of  $A_\Lambda$  is equal to  $A_\Gamma \alpha$  for some  $\alpha \in \mathbb{Z}^n$ .

$\Leftarrow$ ) Any point in  $\Lambda$  can be written  $A_\Lambda \alpha$  for some  $\alpha \in \mathbb{Z}^n$ , which is equal to  $A_\Gamma V \alpha$ , but  $V \alpha \in \mathbb{Z}^n$  so  $A_\Lambda \alpha \in \Gamma$  which is what we needed.

ii) By i), we have some  $V \in \mathbb{Z}^{n \times n}$  such that  $V = A_\Gamma^{-1} A_\Lambda$ . Since  $V$  has only integer elements, we have  $\mathbb{Z} \ni \det(V) = \det(A_\Lambda)/\det(A_\Gamma) \neq 0$ . By considering the relation between a matrix and its cofactor, we get  $U := \det(V)V^{-1} \in GL_n(\mathbb{Z})$ . We get

$$A_\Lambda = A_\Gamma V \Rightarrow A_\Lambda (\det(V)V^{-1}) = A_\Gamma \det(V) \Rightarrow \frac{\det(A_\Lambda)}{\det(A_\Gamma)} A_\Gamma = A_\Lambda U,$$

and therefore we are done by i).

iii) By ii) we have that precisely that  $k\Gamma \subseteq \Lambda$  for some  $0 \neq k \in \mathbb{Z}$ , but then  $k\gamma \in \Lambda$  for any  $\gamma \in \Gamma$  which is enough.  $\odot$

**Lemma A.0.6.** The product of lattices  $\Gamma_1 \times \cdots \times \Gamma_m$  is congruent to  $\Gamma_{\sigma(1)} \times \cdots \times \Gamma_{\sigma(m)}$  for any  $\sigma \in S_m$ .

*Proof.* Let  $A_i$  be basis matrices for  $\Gamma_i$  of dimension  $n_i$  respectively, then we have

$$\Gamma_1 \times \cdots \times \Gamma_m = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \mathbb{Z}^{n_1 + \cdots + n_m}.$$

By Lemma 1.4.2, we can re-arrange the rows by left multiplication with an element of  $O_n(\mathbb{R})$  and the columns by right multiplication with an element of  $GL_n(\mathbb{Z})$ . We are done by lemma 1.2.7, since we can find  $C \in O_n(\mathbb{R}), B \in GL_n(\mathbb{Z})$  such that,

$$\begin{bmatrix} A_{\sigma(1)} & 0 & \cdots & 0 \\ 0 & A_{\sigma(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\sigma(m)} \end{bmatrix} = C \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} B. \quad \odot$$



# Appendix B

## Polyhedral Cones & Quadratic Forms

Chapter 4 and 5 describe polyhedral cones and the embedding of pairs of ternary quadratic forms in  $\bar{V} \times \bar{V} \subseteq \mathbb{R}^{12}$ . Recall that  $\bar{V} \times \bar{V}$  and any of its subsets is centrally anti-symmetric. Note that if  $f \in \bar{V}$ , then  $f_{33} \geq f_{ij} \geq 0$  for any of the coordinates  $f_{ij}$  of  $f$ , the intersection of a polyhedral cone  $P_c(A, B) \subset \bar{V} \times \bar{V}$  with the hyperplane  $H := \{(f, g) \in \bar{V} \times \bar{V} : f_{33} + g_{33} = 1\}$  is a bounded convex polytope, from which  $P_c(A, B)$  can be recovered by forming the cone. In this way, we can also apply existing algorithms in a computer program to find edges. However, for reasons explained in section 5.5, we needed to implement our own algorithm using the lemmas of this chapter. The following results except lemma B.0.8 are given by Schiemann [2] and follow from [26] chapters 2 and 3. We say that a vector  $k$  represent an edge of a polyhedral cone if  $k\mathbb{R}_0^+$  is a 1-face of it. We may refer to both  $k$  and the set  $k\mathbb{R}_0^+$  as an edge of the polyhedral cone. Lastly, all the definitions used in this section can be found in chapter 4.

**Lemma B.0.1.** *The faces of a polyhedral cone are polyhedral cones and faces of a face are faces of the original polyhedral cone.*

*Proof.* Follows from the definitions of faces. ⊙

**Lemma B.0.2.** *Let  $k_1, \dots, k_r$  represent the different 1-faces of a polyhedral cone  $P_c(A, \emptyset)$ . The vectors  $k_i$  form an  $\mathbb{R}_0^+$ -linearly independent set.*

*Proof.* Define  $A_i \subseteq A$  as the sets such that

$$k_i\mathbb{R}_0^+ = P_c(A, \emptyset) \cap \bigcap_{a \in A_i} a^\perp. \quad (\star)$$

Since the edges are distinct,  $A_i$  must be different for each  $i$  and no  $A_i$  is included in another. Assume by contradiction that  $k_i$  are not  $\mathbb{R}_0^+$ -linearly independent. Say without loss of generality that  $k_1 = \lambda_2 k_2 + \dots + \lambda_r k_r$  for  $\lambda_i \geq 0$ , where not all are equal to 0. To each  $k_2, \dots, k_r$ , there is some  $a \in A_1$  such that  $a \cdot k > 0$ . However, this means that  $a \cdot (\lambda_2 k_2 + \dots + \lambda_r k_r) > 0$  for some  $a \in A_1$ , which is a contradiction of  $(\star)$ . ⊙

**Lemma B.0.3.** *Let  $k_1, \dots, k_r$  represent the distinct 1-faces of a centrally anti-symmetric polyhedral cone  $P_c(A, \emptyset)$ . It follows that*

$$P_c(A, \emptyset) = \sum_{i=1}^r k_i\mathbb{R}_0^+.$$

*Sketch of Proof.* It is clear that “ $\subseteq$ ” holds. To see the other inclusion, we argue as follows. Since  $P_c(A, \emptyset)$  is centrally anti-symmetric, we can find a hyperplane  $H$  that does not go through 0, such that  $P_c(A, \emptyset) \cap H$  is bounded. Since each edge  $k_i\mathbb{R}_0^+$  corresponds to a vertex of  $P_c(A, \emptyset) \cap H$  and vice versa (basically since  $k_i\mathbb{R}_0^+$  is not included in this intersection), it is enough to show that any point of bounded convex polytope  $P$  is the convex hull of its 0-faces, meaning its vertices. This is given by theorem 2.4.9 of [26]. ⊙

**Lemma B.0.4.** *Let  $F$  be an  $m$ -face of a centrally anti-symmetric polyhedral cone  $P_c(A, \emptyset)$ , where  $m \geq 1$ . Then,*

$$\dim \bigcap_{a \in A: F \subseteq a^\perp} a^\perp = m.$$

*Proof.* For the “ $\geq$ ” part of the statement, we simply note that obviously

$$F \subseteq \bigcap_{a \in A: F \subseteq a^\perp} a^\perp.$$

To show “ $\leq$ ” we argue as follows. Let  $A' := \{a \in A : F \subseteq a^\perp\}$  and let  $A'' := \{a \in A : F \not\subseteq a^\perp\}$  so that  $A = A' \cup A''$ . We first show that  $A''$  is non-empty. If it were, we would have

$$F \subseteq \bigcap_{a \in A} a^\perp \subseteq P_c(A, \emptyset).$$

Now if  $x \in P_c(A, \emptyset)$ , then  $-x \notin P_c(A, \emptyset)$ . It follows that  $\bigcap_{a \in A} a^\perp = \{0\}$ . This is a contradiction since  $F$  is of dimension  $m \geq 1$ . By lemmas 1.0.2,  $F$  is generated by its edges, say  $k_1, \dots, k_s$ . Since  $A''$  is non-empty, for each  $a \in A''$  there is an edge such that  $a \cdot k > 0$ . Adding all such vectors  $k$  we get a point  $y \in F$  such that  $a \cdot y > 0$  for each  $a \in A''$ . Let  $y_1, \dots, y_r$  be a basis for  $\bigcap_{a \in A'} a^\perp$ . There exists  $\varepsilon_i > 0$  for  $i = 1, \dots, r$  such that  $a \cdot (\varepsilon_i y_i + y) \geq 0 \ \forall a \in A''$ , meaning that  $\varepsilon_i y_i + y \in P_c(A, \emptyset)$ . In particular, since  $a \cdot (\varepsilon_i y_i + y) = 0$  for all  $a \in A'$ , we have  $\varepsilon_i y_i + y \in F$ , implying directly that  $r \leq m$ .  $\odot$

**Corollary B.0.5.** *Let  $k_1 \neq k_2$  represent different edges of a centrally anti-symmetric polyhedral cone  $P_c(A, \emptyset)$ .*

*Let  $\{a_1, \dots, a_r\} =$*

*$\{a \in A : k_1 \subseteq a^\perp\}$  and  $\{a'_1, \dots, a'_s\} = \{a \in A : k_2 \subseteq a^\perp\}$ . The following are equivalent:*

- i)  $k_1 \mathbb{R}_0^+ + k_2 \mathbb{R}_0^+$  is a 2-face of  $P_c(A, \emptyset)$ ,*
- ii)*

$$\dim \bigcap_{a \in \{a_1, \dots, a_r\} \cap \{a'_1, \dots, a'_s\}} a^\perp = 2.$$

*Proof.* Let  $F = k_1 \mathbb{R}_0^+ + k_2 \mathbb{R}_0^+$  and let  $k_i$  represent the edges of  $P_c(A, \emptyset)$ .

*i)  $\Rightarrow$  ii) :* The  $a \in A$  such that  $F \subseteq a^\perp$  are precisely those such that  $k_1, k_2 \subseteq a^\perp$ . This direction now follows by lemma B.0.3.

*ii)  $\Rightarrow$  i) :* By definition,  $F$  is a 2-face if there are elements  $a_i \in A$  such that  $F = (\bigcap_i a_i^\perp) \cap P_c(A, \emptyset)$  and  $F$  is 2-dimensional.  $F$  is 2-dimensional since  $k_1, k_2$  represent different edges, meaning  $k_1 \mathbb{R}_0^+ \neq k_2 \mathbb{R}_0^+$  (note also that  $k_1 \neq -k_2$  since the cone was centrally anti-symmetric). It is clear that

$$F \subseteq P_c(A, \emptyset) \cap \bigcap_{a \in \{a_1, \dots, a_r\} \cap \{a'_1, \dots, a'_s\}} a^\perp. \quad (\#)$$

The right hand side of (#) is at least 2-dimensional since  $F$  is, therefore due to the assumption, the right hand side is precisely 2-dimensional. We are done by definition if we can show that the inclusion of (#) is in fact an equality. If this were not the case, then we would find an  $x \neq 0$  in the right hand side of (#) such that  $x \notin F$ . We may choose  $x$  to be either equal to  $k_1 - \epsilon k_2$  or  $k_2 - \epsilon k_1$  for some small  $\epsilon > 0$  by convexity. By virtue of being in  $P_c(A, \emptyset)$ , we can write  $x = \sum \lambda_i k_i$  for  $\lambda_i \geq 0$ . Since both  $x + \epsilon k_1, x + \epsilon k_2$  are  $\mathbb{R}_0^+$ -linear combinations of the edges  $k_i$  and one is equal to either  $k_1$  or  $k_2$ , this is a contradiction of lemma B.0.2.  $\odot$

**Lemma B.0.6.** *Let  $\dim P_c(A, \emptyset) = r, v \neq 0$  and  $a \in A$ . We have*

- 1)  $\dim(P_c(A, \emptyset) \cap a^\perp) < r - 1 \Rightarrow P_c(A \setminus \{a\}, \emptyset) = P_c(A, \emptyset)$ .*
- 2)  $\exists x \in P_c(A, \emptyset) : x \cdot v > 0 \Rightarrow \dim(P_c(A \cup \{v\}, \emptyset)) = r$ .*

*Proof.*

1) We show the negation. Let  $x \in P_c(A \setminus \{a\}, \emptyset) \setminus P_c(A, \emptyset)$ , in other words  $x \cdot a < 0$ . Let  $y_1, \dots, y_r \in P_c(A, \emptyset)$  be linearly independent and without loss of generality let also  $x, y_2, \dots, y_r$  be linearly independent. The dimension of  $P_c(A, \emptyset) \cap a^\perp$  must now be equal to  $r$  since for  $i = 2, \dots, r$  there is an  $\epsilon_i \geq 0$  with

$$\epsilon_i x + y_i \in P_c(A \setminus \{a\}, \emptyset) \cap a^\perp = P_c(A, \emptyset) \cap a^\perp.$$

2) Follows analogously. ◻

**Lemma B.0.7.** *Let  $P_c(A, \emptyset)$  be a centrally anti-symmetric polyhedral cone and let  $K$  be the set of its edges and  $L$  the set of its 2-faces. Let  $v \neq 0$ . We have for the set  $K'$  of edges of  $P_c(A \cup \{v\}, \emptyset)$ :*

$$\{k\mathbb{R}_0^+ \in K'\} = \{k\mathbb{R}_0^+ \in K : k \cdot v \geq 0\} \dot{\cup} \{F \cap v^\perp : F = k_1\mathbb{R}_0^+ + k_2\mathbb{R}_0^+ \in L \ \& \ k_1 \cdot v > 0, k_2 \cdot v < 0\}.$$

We define for the sake of notation the following three sets,  $\mathcal{K}_1 := \{k\mathbb{R}_0^+ \in K' : k \cdot v = 0\}$ ,  $\mathcal{K}_2 := \{k\mathbb{R}_0^+ \in K : k \cdot v = 0\}$  and  $\mathcal{K}_3 := \{F \cap v^\perp : F = k_1\mathbb{R}_0^+ + k_2\mathbb{R}_0^+ \in L \ \& \ k_1 \cdot v > 0, k_2 \cdot v < 0\}$ .

*Proof.* We first show  $\{k\mathbb{R}_0^+ \in K' : k \cdot v > 0\} = \{k\mathbb{R}_0^+ \in K : k \cdot v > 0\}$ . For  $k \in K$  with  $k \cdot v > 0$  we have  $k \in P_c(A \cup \{v\}, \emptyset)$  and by definition,  $k$  represents an edge of  $P_c(A \cup \{v\}, \emptyset)$  precisely when it is a 1-dimensional intersection of  $P_c(A \cup \{v\}, \emptyset)$  and some of its supporting hyperplanes. Now since  $k \notin v^\perp$ , the supporting hyperplanes we have left to choose from are those corresponding to  $A$ , which is precisely the case for edges in  $P_c(A, \emptyset)$ .

We are left to show that  $\mathcal{K}_1 = \mathcal{K}_2 \dot{\cup} \mathcal{K}_3$ . Similarly as in the first part of the proof we have  $\mathcal{K}_1 \supseteq \mathcal{K}_2$ . To see that  $\mathcal{K}_1 \supseteq \mathcal{K}_3$ , let  $F \in L$  and  $F \cap v^\perp \in \mathcal{K}_3$ . By definition we have for some  $A' \subseteq A$  we have

$$\begin{aligned} F &= \bigcap_{a \in A} a^\perp \cap P_c(A, \emptyset) \Rightarrow \\ \Rightarrow F \cap v^\perp &= \bigcap_{c \in A \cup \{v\}} c^\perp \cap P_c(A \cup \{v\}, \emptyset). \end{aligned}$$

It follows that  $F \cap v^\perp \in \mathcal{K}_1$ , meaning it is an edge of  $P_c(A \cup \{v\}, \emptyset)$ . This is because  $\dim(F \cap v^\perp) = 1$ , which follows from the fact that if  $F = k_1\mathbb{R}_0^+ + k_2\mathbb{R}_0^+$  and  $v \cdot (\lambda_1 k_1 + \lambda_2 k_2) = 0$ , then  $\lambda_2$  is determined uniquely from  $\lambda_1$ . Explicitly,  $F \cap v^\perp = (|k_2 \cdot v|k_1 + |k_1 \cdot v|k_2)\mathbb{R}_0^+$ .

We are left to show that  $\mathcal{K}_1 \subseteq \mathcal{K}_2 \dot{\cup} \mathcal{K}_3$ . Let  $k \in \mathcal{K}_1$ . First we define the vector space  $U$  and make an observation due to lemma B.0.4.

$$U := \bigcap_{c \in A : c \cdot k = 0} c^\perp \quad \& \quad \dim \bigcap_{c \in A \cup \{v\} : c \cdot k = 0} c^\perp = 1.$$

We see that  $\dim(U) \geq 1$  and that  $k \in U \setminus \{0\}$ . Let  $y_1 \in U \setminus \{0\}$  be any vector such that  $v \cdot y_1 = 0$ . Assume that there are at least three linearly independent vectors  $y_1, y_2, y_3$  in  $U$ , meaning  $\dim(U) \geq 3$ . Since

$$U \cap v^\perp = \bigcap_{c \in A \cup \{v\} : c \cdot k = 0} c^\perp,$$

we have that  $v \cdot y_i = 0$  if and only if  $i = 1$ . But since  $v \cdot y_i \neq 0$  for  $i = 2, 3$ , have for some  $s, t \in \mathbb{R} \setminus \{0\}$  that  $sy_2 + ty_3 = 0$ . However, this means that  $sy_2 + ty_3 \in U \cap v^\perp$  and therefore  $\dim(U \cap v^\perp) \geq 2$  which is a contradiction. We have shown that  $\dim(U) \in \{1, 2\}$ . Note that  $k \in U \cap P_c(A, \emptyset)$ . As a direct consequence,

$$\dim \bigcap_{c \in A : c \cdot k = 0} c^\perp \cap P_c(A, \emptyset) \in \{1, 2\}. \quad (\square)$$

**Case 1:**  $(\square)$  is equal to 1. This means

$$\bigcap_{c \in A : c \cdot k = 0} c^\perp \cap P_c(A, \emptyset) = k\mathbb{R}_0^+,$$

and therefore  $k \in \mathcal{K}_2$ .

**Case 2:**  $(\square)$  is equal to 2. This means

$$F := \bigcap_{c \in A: c \cdot k = 0} c^\perp \cap P_c(A, \emptyset) \in L,$$

and  $k\mathbb{R}_0^+$  is not an edge of  $P_c(A, \emptyset)$ , which can be seen by lemma B.0.4 and the fact that  $\dim(U) \neq 1$ . This together with lemma B.0.3 says that  $F = k_1\mathbb{R}_0^+ + k_2\mathbb{R}_0^+$  for two edges  $k_1, k_2 \in K$  that represent distinct edges from  $k$  (since  $k\mathbb{R}_0^+$  was not an edge). However,  $F \cap v^\perp$  is at most of dimension 1 by what we have previously done and therefore  $k \in F \cap v^\perp$  implies that the dimension is in fact 1. It follows that  $v \cdot k_1 \neq 0, v \cdot k_2 \neq 0$  and that without loss of generality  $v \cdot k_1 > 0$  and  $v \cdot k_2 < 0$ . Finally,  $F \cap v^\perp = k\mathbb{R}_0^+ \in \mathcal{K}_3$ . The disjointedness of the union follows from the disjointedness of case 1 and 2.  $\odot$

**Lemma B.0.8.** *Let  $k_i$  be the edges of  $P_c(A, \emptyset)$  and  $k'_j$  the edges of  $P_c(C, \emptyset)$ . The edges of  $P_c(A, \emptyset) \times P_c(C, \emptyset)$  are  $k_i \times \{0\}$  and  $\{0\} \times k'_j$ .*

*Proof.* An edge of  $P_c(A, \emptyset) \times P_c(C, \emptyset)$  is a 1-dimensional intersection on the form

$$k\mathbb{R}_0^+ = \bigcap_{a \in A, c \in C} (a, 0)^\perp \cap (0, c)^\perp \cap \left( P_c(A, \emptyset) \times P_c(C, \emptyset) \right) = \left( \bigcap_{a \in A} a^\perp \cap P_c(A, \emptyset) \right) \times \left( \bigcap_{c \in C} c^\perp \cap P_c(C, \emptyset) \right).$$

Now for the right hand side to be 1-dimensional, exactly one of the terms in the cartesian product is 1-dimensional and the other is 0-dimensional, meaning it is equal to  $\{0\}$ . The statement now follows.  $\odot$

# Appendix C

## Schiemann's Theorem III

First, let's recall the constraints that define a sign reduced form  $f$ . We reformulate them in a different for the sake of proposition C.0.3.

1a)  $f$  is Minkowski reduced,

1b)  $f_{12} \geq 0, f_{13} \geq 0,$

1c)  $2f_{23} > -f_{22},$

and the following boundary conditions hold:

2a)  $f_{12} = 0 \implies f_{23} \geq 0,$

2b)  $f_{13} = 0 \implies f_{23} \geq 0,$

3a)  $f_{11} = f_{22} \implies |f_{23}| \leq f_{13},$

3b)  $f_{22} = f_{33} \implies f_{13} \leq f_{12},$

4a)  $f_{11} + f_{22} - 2f_{12} - 2f_{13} + 2f_{23} = 0 \implies f_{11} - 2f_{13} - f_{12} \leq 0,$

4b)  $2f_{12} = f_{11} \implies f_{13} \leq 2f_{23},$

4c)  $2f_{13} = f_{11} \implies f_{12} \leq 2f_{23},$

4d)  $2f_{23} = f_{22} \implies f_{12} \leq 2f_{13}.$

**Definition C.0.1.** A ternary positive definite form  $g$  is called Eisenstein reduced if

a)  $g$  is Minkowski reduced,

b) Either  $g_{12}, g_{13}, g_{23} > 0$  (then  $g$  is called positive), or  $g_{12}, g_{13}, g_{23} \leq 0$  (then  $g$  is called non-negative),

c)  $g_{11} = g_{22} \implies |g_{23}| \leq |g_{13}|$  and  $g_{22} = g_{33} \implies |g_{13}| \leq |g_{12}|,$

d) 1) If  $g_{12}, g_{13}, g_{23} > 0,$  then

$$2g_{12} = g_{11} \implies g_{13} \leq 2g_{23},$$

$$2g_{13} = g_{11} \implies g_{12} \leq 2g_{23},$$

$$2g_{23} = g_{22} \implies g_{12} \leq 2g_{13}.$$

2) If  $g_{12}, g_{13}, g_{23} \leq 0,$  then

$$g_{11} + g_{22} + 2g_{12} + 2g_{13} + 2g_{23} = 0 \implies g_{11} + 2g_{13} + g_{12} \leq 0,$$

$$2g_{12} = -g_{11} \implies g_{13} = 0,$$

$$2g_{13} = -g_{11} \implies g_{12} = 0,$$

$$2g_{23} = -g_{22} \implies g_{12} = 0,$$

**Theorem C.0.2.** To each ternary form  $f,$  there is a unique equivalent representative that is Eisenstein reduced.

Theorem C.0.2 is for example stated in [27, p. 188] and in [28].

**Proposition C.0.3.** *To each ternary positive definite quadratic form  $q$  there is a unique equivalent form in  $V$ .*

The following deduction is skipped in Schiemann's thesis, but as we shall see, the proof is just a sequence of somewhat tedious calculations.

*Proof.* We apply theorem C.0.2 by finding a bijection  $\psi$  from the set of Eisenstein forms to the set of sign reduced forms.

$$\psi(f) := \begin{cases} f \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix} & \text{if } f \text{ is positive,} \\ f \circ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_{11} & -f_{12} & -f_{13} \\ -f_{12} & f_{22} & f_{23} \\ -f_{13} & f_{23} & f_{33} \end{pmatrix} & \text{if } f \text{ is non-positive and} \\ & \text{\scriptsize } f_{12} \neq 0 \text{ and } f_{13} \neq 0, \\ f \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} f_{11} & 0 & -f_{13} \\ 0 & f_{22} & -f_{23} \\ -f_{13} & -f_{23} & f_{33} \end{pmatrix} & \text{if } f \text{ is non-positive and} \\ & \text{\scriptsize } f_{12} = 0, \\ f \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_{11} & -f_{12} & 0 \\ -f_{12} & f_{22} & -f_{23} \\ 0 & -f_{23} & f_{33} \end{pmatrix} & \text{if } f \text{ is non-positive and} \\ & \text{\scriptsize } f_{12} \neq 0 \text{ and } f_{13} = 0. \end{cases}$$

Note first that  $\psi(f)$  is equivalent to  $f$  for each Eisenstein reduced form and therefore the image of  $\psi$  contains a unique representative of each ternary positive definite form. It follows that  $\psi$  is injective. We argue that  $f$  is surjective as follows. We can decompose the set of sign reduced forms  $V$  into four disjoint sets as follows,

$$\begin{aligned} U_1 &:= \{f \in V : f_{12}, f_{13}, f_{23} > 0\}, & U_3 &:= \{f \in V : f_{12} = 0\}, \\ U_2 &:= \{f \in V : f_{12}, f_{13} > 0 \text{ and } f_{23} \leq 0\}, & U_4 &:= \{f \in V : f_{12} > 0, f_{13} = 0\}. \end{aligned}$$

Finally, we show that each  $U_i$  lies in the image of  $\psi$ ,

**Case  $U_1$ :** It is straightforward to see that each element of  $U_1$  is a positive Eisenstein reduced form, just by looking at the definition of  $V$ . Therefore, for each  $f \in U_1$  we have  $\psi(f) = f$ , which means that  $U_1$  is in the image of  $\psi$ .

**Case  $U_2$ :** If  $f \in U_2$ , it suffices to show that

$$g := \begin{pmatrix} f_{11} & -f_{12} & -f_{13} \\ -f_{12} & f_{22} & f_{23} \\ -f_{13} & f_{23} & f_{33} \end{pmatrix}$$

is Eisenstein reduced, since then  $\psi(g) = f$ . We note that  $g_{12}, g_{13}, g_{23} \leq 0$ . As seen toward the end of section 1.3,  $g$  is Minkowski reduced. We note that  $g_{11} + g_{22} + 2g_{12} + 2g_{13} + 2g_{23} = 0$  implies  $g_{11} + 2g_{13} + g_{12} \leq 0$  directly since  $f$  is sign reduced. It is clear that  $g$  satisfies condition c) of Eisenstein reduction. Finally, by the fact that  $f$  is sign reduced and is in  $U_2$ ,

$$\begin{aligned} 2g_{12} = -g_{11} &\Rightarrow 2f_{12} = f_{11} \Rightarrow 0 < f_{13} \leq 2f_{23} \leq 0 \\ 2g_{13} = -g_{11} &\Rightarrow 2f_{13} = f_{11} \Rightarrow 0 < f_{12} \leq 2f_{23} \leq 0 \\ 2g_{23} = -g_{22} &\Rightarrow 2f_{23} = f_{22} \Rightarrow 0 < f_{23} \leq 0 \end{aligned}$$

In each case we have a contradiction, meaning that  $2g_{12} \neq -g_{11}, 2g_{13} \neq -g_{11}, 2g_{23} \neq -g_{22}$ . We have shown that  $g$  satisfies all criteria to be a Eisenstein reduced form, which is enough.

**Case  $U_3$ :** If  $f \in U_3$ , it suffices to show that

$$g := \begin{pmatrix} f_{11} & 0 & -f_{13} \\ 0 & f_{22} & -f_{23} \\ -f_{13} & -f_{23} & f_{33} \end{pmatrix}$$

is Eisenstein reduced, since then  $\psi(g) = f$ . As noted toward the end of section 1.3,  $g$  is Minkowski reduced. We have  $g_{12}, g_{13}, g_{23} \leq 0$ . Condition c) for Eisenstein reduced forms follows directly. Further,  $g_{11} + g_{22} + 2g_{12} + 2g_{13} + 2g_{23} = 0$  implies  $f_{11} + f_{22} - 2f_{13} - 2f_{23} = 0$  which means  $f_{11} = 2f_{13}$  by the Minkowski reduction of  $f$ . As a consequence  $g_{11} + 2g_{13} + g_{12} = g_{11} + 2g_{13} = 0$ . Finally we note that  $0 = g_{12} \neq -g_{11}$  and in particular since  $g_{12} = 0$ ,  $g$  satisfies all conditions to be a Eisenstein reduced form.

**Case  $U_4$ :** If  $f \in U_4$ , it suffices to show that

$$g := \begin{pmatrix} f_{11} & -f_{12} & 0 \\ -f_{12} & f_{22} & -f_{23} \\ 0 & -f_{23} & f_{33} \end{pmatrix}$$

is Eisenstein reduced, since then  $\psi(g) = f$ . As noted toward the end of section 1.3,  $g$  is Minkowski reduced. We have  $g_{12}, g_{13}, g_{23} \leq 0$ . Condition  $c$ ) for Eisenstein reduced forms follows directly. Further,  $g_{11} + g_{22} + 2g_{12} + 2g_{13} + 2g_{23} = 0$  implies  $f_{11} + f_{22} - 2f_{12} - 2f_{23} = 0$  and as a consequence  $2f_{23} = f_{22}$  by the Minkowski reduction of  $f$ . This implies  $0 < f_{12} \leq f_{13} = 0$  which is a contradiction. Since  $g_{12} \neq 0$  and  $g_{13} = 0$ , we only need to check that  $2g_{13} \neq -g_{11}$  and  $2g_{23} \neq -g_{22}$ . Assuming otherwise, we get the following contradictions since  $f$  is sign reduced and is in  $U_4$ ,

$$\begin{aligned} 2g_{13} = -g_{11} &\Rightarrow 2f_{13} = f_{11} \Rightarrow 0 < f_{13} \\ 2g_{23} = -g_{22} &\Rightarrow 2f_{23} = f_{22} \Rightarrow 0 < f_{12} \leq 2f_{13} = 0. \end{aligned} \quad \ominus$$

**Lemma C.0.4.** *The closure of  $V$  is precisely given by the following system of inequalities,*

$$\begin{cases} 0 \leq q_{11} \leq q_{22} \leq q_{33} \\ 0 \leq 2q_{12} \leq q_{11} \ \& \ 0 \leq 2q_{13} \leq q_{11} \\ -q_{22} \leq 2q_{23} \leq q_{22} \\ q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23} \geq 0 \end{cases} \quad (\text{C.1})$$

*Proof.* We first define  $W := \{q \in \mathbb{R}^6 : q \text{ satisfies (C.1)}\}$ . We aim to prove  $\overline{V} = W$ . Clearly,  $W$  is a closed set and  $V \subseteq W$ , showing  $\overline{V} \subseteq W$ . Now take any  $q \in W$ . We find a sequence  $q_k \in V$  such that  $q_k \rightarrow q$ . We have that  $q$  satisfies all but the first of the conditions for Minkowski reduction as in theorem 4.3.3, among the other rows of the system of inequalities, only row 6, 8 and 9 do not follow directly. To see how we get row 6, note that  $0 \leq q_{11} - 2q_{13}, 0 \leq q_{22} - 2q_{23}$  and  $0 \leq q_{12}$  implies  $0 \leq q_{11} + q_{22} + 2q_{12} - 2q_{13} - 2q_{23}$ . The other follows similarly. Now we adjust  $q$  slightly so that it satisfies  $q_{11} > 0, 1c), 2a, b), 3a, b)$  and  $4a, b, c, d)$ . We define a sequence of a quadratic forms on the form

$$I^k = \begin{bmatrix} \epsilon_{11}^k & \epsilon_{12}^k & \epsilon_{13}^k \\ \epsilon_{21}^k & \epsilon_{22}^k & \epsilon_{23}^k \\ \epsilon_{31}^k & \epsilon_{32}^k & \epsilon_{33}^k \end{bmatrix},$$

such that  $I^k \rightarrow 0$  and  $q + I^k \in V$  for all  $k$ . For this purpose, we look at the following example,

$$I^k = \begin{bmatrix} 1/k & \frac{1}{2k^2} & \frac{1}{2k^3} \\ \frac{1}{2k^2} & 2/k & \frac{1}{2k^4} \\ \frac{1}{2k^3} & \frac{1}{2k^4} & 3/k \end{bmatrix}.$$

It is not hard to see that  $q + I^k$  are Minkowski reduced forms for big  $k$ , by theorem 4.3.3, and  $1b, c)$  also hold for  $q + I^k$ . For the boundary condition of sign reduced forms, we argue that for big enough  $k$ , none of the assumptions of the boundary conditions for sign reduction are satisfied, implying that the boundary conditions hold. As an example we look at  $4a)$ . We want the following to be true,

$$q_{11} + q_{22} - 2q_{12} - 2q_{13} + 2q_{23} + 3/k - 1/k^2 - 1/k^3 + 1/k^4 = 0 \Rightarrow q_{11} - 2q_{13} - q_{12} + 1/k - 1/k^3 - 1/(2k^2) \leq 0.$$

Now the left hand side can only be satisfied for at most a finite number of values  $k$ , since  $q$  is fixed. The other conditions work similarly. In other words,  $q + I^k$  satisfy all the boundary conditions for all big  $k$ , implying  $q + I^k \in V$  and  $q \in \overline{V}$ .  $\ominus$

**Lemma C.0.5.** *Let  $T = P_c(A, B) \subseteq \overline{V} \times \overline{V}$ . We have  $\overline{T \cap (V \times V)} = \overline{T_{V \times V}}$ .*

*Proof.*

“ $\subseteq$ ”: By construction,  $T \cap (V \times V) \subseteq T_i$  for each  $i$  which is enough.

“ $\supseteq$ ”: If  $T = \emptyset$  or  $T = \{0\}$ , then the equality is trivial. Otherwise we argue as follows after noting that since  $T \subseteq \overline{V} \times \overline{V}$ , we have for any pair  $(q_1, q_2) \in T$  that  $q_1 \cdot c \geq 0$  and  $q_2 \cdot c \geq 0$  for any  $c$  such that  $(c, d) \in \mathcal{C}$  for some

d. Consider  $(f, g) \in \overline{T_{V \times V}} \subseteq T$ . Assume  $(f_j, g_j) \rightarrow (f, g)$  with  $(f_j, g_j) \in T_{V \times V}$  and  $\{(c_1, d_1), \dots, (c_r, d_r)\} = \{(c, d) \in \mathfrak{C} : T_{V \times V} \not\subseteq (c, 0)^\perp\}$  and  $\{(c'_1, d'_1), \dots, (c'_s, d'_s)\} = \{(c, d) \in \mathfrak{C} : T_{V \times V} \not\subseteq (0, c)^\perp\}$ . Let  $(p_i, q_i) \in T_{V \times V}$  be such that  $(p_i, q_i) \cdot (c_i, 0) > 0$  for  $i = 1, \dots, r$  and  $(p'_i, q'_i) \in T_{V \times V}$  be such that  $(p'_i, q'_i) \cdot (0, c_i) > 0$  for  $i = 1, \dots, s$ . Now we define for  $\lambda > 0$  and  $j \in \mathbb{N}$

$$h(\lambda, j) := (f_j, g_j) + \lambda \left( \sum_{i=0}^r (p_i, q_i) + \sum_{i=0}^s (p'_i, q'_i) \right).$$

Because of

$$h(\lambda, j) \in T \cap (V \times V) \text{ and } h(\lambda, j) \rightarrow (f, g),$$

where the limit considers  $\lambda \rightarrow 0$  and  $j \rightarrow \infty$ , the claim of the proposition follows.  $\odot$

The polynomial that correspond to elements of  $M$  are now listed. It is clear that all elements of  $M_3$  are positive definite, the lower block matrices of elements in  $M_1, M_2$  are also positive definite.

$$\begin{aligned} x^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x &= x_3^2, \\ x^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} x &= 2x_2^2 + 2x_3^2 \pm x_2x_3 = 2(x_2 \pm x_3/2)^2 + \frac{1}{2}x_3^2, \\ x^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} x &= 2x_1^2 + 2x_2^2 + 2x_3^2 \pm 2x_2x_3 = 2x_1^2 + 2(x_2 \pm \frac{1}{2}x_3)^2 + \frac{3}{2}x_3^2, \\ x^T \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & \pm 1 \\ 1 & \pm 1 & 2 \end{pmatrix} x &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 \pm 2x_2x_3 = 2(x_1 + \frac{1}{2}x_3)^2 + 2(x_2 \pm \frac{1}{2}x_3)^2 + x_3^2, \\ x^T \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \pm 1 \\ 0 & \pm 1 & 2 \end{pmatrix} x &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 \pm 2x_2x_3 = 2(x_1 + \frac{1}{2}x_2)^2 + 2(x_3 \pm \frac{1}{2}x_2)^2 + x_2^2, \\ x^T \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} x &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 = 2(x_2 + \frac{1}{2}x_1)^2 + 2(x_3 + \frac{1}{2}x_1)^2 + x_1^2, \\ x^T \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} x &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 2(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3)^2 + \frac{3}{2}(x_2 + \frac{1}{3}x_3)^2 + \frac{4}{3}x_3^2. \end{aligned}$$



# Appendix D

## Code

We now present the code in the language of Julia with which we verified Schiemann's theorem. It consists of 9 different files. They follow the algorithms from section 4.4 and chapter 5 closely. Observe that  $P_c(\mathfrak{A}, \mathfrak{B}) \cap \overline{V} = P_c(\emptyset, \mathfrak{B}) \cap \overline{V}$ , and therefore it is not hard to see that  $(P_c(\mathfrak{A}, \mathfrak{B}) \times P_c(\mathfrak{A}, \mathfrak{B})) \cap (\overline{V} \times \overline{V})$  is equal to  $(\overline{V} \times \overline{V})_{V \times V}$ . We make use of this when defining  $T$  in MainProg.jl, which is the set  $(\overline{V} \times \overline{V})_{V \times V}$ . Further, when applying the algorithm to find  $T_{V \times V}$  for  $T \in \mathcal{T}_i$  with  $i \geq 1$ , we do not need to intersect with  $P_c(\mathfrak{A}, \mathfrak{B}) \cap P_c(\mathfrak{A}, \mathfrak{B})$  since we already did it for  $\mathcal{T}_0$ . Finally, we find the edges of  $(\overline{V} \times \overline{V})_{V \times V}$  to be the set  $M \times \{0\} \cup \{0\} \times M$  in the natural way by lemma B.0.8. As we mentioned, when we used only 1 processor, the program took about 3 hours. When we used 50 processors it took only 19 minutes. The computer we used has the following properties and packages,

**CPU:** Intel(R) Xeon(R) Platinum 8180 CPU @ 2.50 Ghz,

**OS:** Fedora 32, **Packages:** Abstract Algebra v0.9.0, Nemo v0.17.0 & Hecke v.0.8.0.

We use Hecke in order to utilize `fmpz_mat` and `fmpq_mat`. `fmpz_mat` is a matrix type of integers and `fmpq_mat` is a matrix type of rational numbers. They are faster than Julia's built-in matrix types. We note that the function `hcat` adds columns to matrices and elements to arrays. Similarly, `vcat` add rows to matrices and elements to arrays, but it does not add empty elements to arrays which is useful. Before we list the files that we used, we mention that one benefit of Julia is that we can have many functions of the same name, but with different types of inputs. We use this frequently in our program. We write in each file what functions we use from the files that we include.

Listing D.1: MainProg.jl

```
1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7 import Base.convert
8 import Dates
9
10 include("VariDeclared.jl") # Uses: PolyCone, PolyConeInfo, Tcover, closV, B, EdgesTemp.
11 include("MainFunc.jl") # Uses: refine.
12 include("PolyOp.jl") # Uses: Times, RowsToArr.
13
14 # #-#-# #
15
16 T=PolyCone(vcat(B,Times(closV,closV)),B);
17 # This is the set  $[\overline{V}] \times [\overline{V}]_{V \times V}$  as in  $\mathcal{T}_0$ .
18 # We made sure to let  $B \subseteq A$  as in section 4.4.
19 E=RowsToArr(Times(EdgesTemp,EdgesTemp)); # The edges of  $\overline{V} \times \overline{V}$ .
20 T=PolyConeInfo(T,E,fmpz(12));
21 # One can check using the edges that the dimension is 12.
```

```

22 Tstart=Tcover(T,fmpz(0),fmpz(0),fmpz_mat[]::Vector{fmpz_mat},fmpz_mat[]::Vector{fmpz_mat});
23 covs=[Tstart]::Vector{Tcover}; # covs corresponds to \mathcal{T}_i in section 5.2.
24 counter=0;
25
26 println(Dates.Time(Dates.now())) # Keeps track of time.
27 while !isempty(covs) # We stop when all elements of the covering is in Delta.
28     covsNew=Tcover[]::Vector{Tcover}
29     global covs
30
31     for cov in covs
32         covsNew=vcat(covsNew,refine(cov))
33     end
34     covs=covsNew
35
36     global counter+=1
37     display(counter)
38     display(length(covs))
39     println(Dates.Time(Dates.now()))
40 end

```

Listing D.2: MainFunc.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7 import Base.convert
8
9 include("VariDeclared.jl") # Uses: Tcover.
10 include("CalcVcrossV.jl") # Uses: VcrossV.
11 include("CalcMIN.jl") # Uses: MIN.
12 include("CalcK.jl") # Uses: K, Tequal.
13 include("CalcLamNxy.jl") # Uses: FinaMaxR.
14 include("PolyOp.jl") # Uses: Times.
15 include("CalcInter.jl") # Uses: InterInfo, CheckPolyIncluded.
16
17 # #-#-# #
18
19 function refine(Tcov::Tcover)
20     Y=Tcov[]::Vector{Tcover} # We return this vector of elements of a covering.
21     minX=MIN(Tcov.Lam,Tcov.X) # We go through minX and minY in the for-loop.
22     minY=MIN(Tcov.Lam,Tcov.Y)
23
24     if !CheckPolyIncluded(Txy,Delta) # Check if in \Delta, then we never update it again.
25         for i in 1:ncols(minX)
26             for j in 1:ncols(minY)
27                 X_xy=vcat(Tcov.X,[minX[:,i]])
28                 Y_xy=vcat(Tcov.Y,[minY[:,j]])
29
30                 K_ref=Times(K(Tcov.Lam,X_xy),K(Tcov.Lam,Y_xy))
31                 Txy=VcrossV(InterInfo(Tcov.T,vcat(K_ref,Tequal(minX[:,i],minY[:,j]))))
32
33                 if Txy!="empty" # It makes sense to stop if the cone is empty.
34                     if CheckPolyIncluded(Txy,Delta2) # We find a new value of \Lambda.
35                         Lamxy=fmpz(2)
36                     elseif CheckPolyIncluded(Txy,Delta1)
37                         Lamxy=fmpz(1)

```

```

38     else
39         Lamxy=fmpz(0)
40     end
41
42     if Lamxy==Tcov.Lam
43         k_xy=Tcov.k+1
44     else
45         k_xy=FindMaxR(Lamxy,X_xy,Y_xy) # Sort out as in case 2.1.
46         X_xy=X_xy[1:k_xy]
47         Y_xy=Y_xy[1:k_xy]
48     end
49     Y=vcat(Y,Tcover(Txy,Lamxy,k_xy,X_xy,Y_xy))
50
51     end
52     end
53     end
54     end
55     return Y
56
57 end

```

Listing D.3: VariDeclared.jl

```

1  using Hecke
2  using Base: IO
3  import Base: show
4  import Base: ==
5  import Base: <=
6  import Base.gcd
7
8  struct PolyCone # Corresponds to a polyhedral cone P_c.
9      A::fmpz_mat
10     B::fmpz_mat
11 end
12
13 struct PolyConeInfo # A polyhedral cone with known edges and dimension.
14     P::PolyCone
15     E::Vector{fmpz_mat}
16     dim::fmpz
17 end
18
19 struct Tcover # An element of a covering as in definition 5.2.4.
20     T::PolyConeInfo
21     Lam::fmpz
22     k::fmpz
23     X::Vector{fmpz_mat}
24     Y::Vector{fmpz_mat}
25 end
26
27 # #-#-# #
28
29 # closV corresponds to the constraints of  $\overline{V}$ .
30 closV=matrix(ZZ,[1 0 0 0 0 0;-1 1 0 0 0 0; 0 -1 1 0 0 0;
31               0 0 0 1 0 0;0 0 0 0 1 0;1 0 0 -2 0 0;
32               1 0 0 0 -2 0;0 1 0 0 0 2;0 1 0 0 0 -2;
33               1 1 0 -2 -2 2]);
34
35 # B corresponds to the strict inequalities that we always have for elements in a covering.
36 B=matrix(ZZ,[1 0 0 0 0 0 0 0 0 0 0 0;0 1 0 0 0 2 0 0 0 0 0 0;

```

```

37         0 0 0 0 0 0 1 0 0 0 0 0;0 0 0 0 0 0 0 1 0 0 0 2])
38
39 # EdgesTemp corresponds to the elements of M.
40 EdgesTemp=matrix(QQ,[0 0 1 0 0 0;0 2 2 0 0 -1;0 2 2 0 0 1;2 2 2 0 0 -1;
41         2 2 2 0 0 1;2 2 2 0 1 -1;2 2 2 0 1 1;2 2 2 1 0 -1;
42         2 2 2 1 0 1;2 2 2 1 1 0;2 2 2 1 1 1])
43
44 # Cbound and Dbound correspond to the set  $\frac{C}{D}$  in section 5.2.1.
45 Cbound=matrix(ZZ,[0 0 0 1 0 0;0 0 0 0 1 0;1 -1 0 0 0 0;
46         1 -1 0 0 0 0;0 1 -1 0 0 0;1 1 0 -2 -2 2;
47         1 0 0 -2 0 0;1 0 0 0 -2 0;0 1 0 0 0 -2]);
48
49 Dbound=matrix(ZZ,[0 0 0 0 1;0 0 0 0 1;0 0 0 1 -1;
50         0 0 0 1 1;0 0 0 1 -1 0;-1 0 0 1 2 0;
51         0 0 0 -1 2;0 0 0 -1 0 2;0 0 0 -1 2 0]);
52
53 # Delta corresponds to  $\Delta$  and  $\Lambda_1, \Lambda_2$  as in section 5.5.
54 Delta= matrix(ZZ,[-1 0 0 0 0 0 1 0 0 0 0 0;1 0 0 0 0 0 -1 0 0 0 0 0;
55         0 -1 0 0 0 0 0 1 0 0 0 0;0 1 0 0 0 0 0 -1 0 0 0 0;
56         0 0 -1 0 0 0 0 1 0 0 0 0;0 0 1 0 0 0 0 0 -1 0 0 0;
57         0 0 0 -1 0 0 0 0 1 0 0 0;0 0 0 1 0 0 0 0 0 -1 0 0;
58         0 0 0 0 -1 0 0 0 0 1 0 0;0 0 0 0 1 0 0 0 0 0 -1 0;
59         0 0 0 0 0 -1 0 0 0 0 0 1;0 0 0 0 0 1 0 0 0 0 0 -1]);
60
61 Lambda1=matrix(ZZ,[-1 0 0 0 0 0 1 0 0 0 0 0;1 0 0 0 0 0 -1 0 0 0 0 0;
62         0 -1 0 0 0 0 0 1 0 0 0 0;0 1 0 0 0 0 0 -1 0 0 0 0;
63         0 0 0 -1 0 0 0 0 1 0 0 0;0 0 0 1 0 0 0 0 0 -1 0 0]);
64
65 Lambda2=matrix(ZZ,[-1 0 0 0 0 0 1 0 0 0 0 0;1 0 0 0 0 0 -1 0 0 0 0 0;
66         0 -1 0 0 0 0 0 1 0 0 0 0;0 1 0 0 0 0 0 -1 0 0 0 0;
67         0 0 -1 0 0 0 0 1 0 0 0 0;0 0 1 0 0 0 0 0 -1 0 0 0;
68         0 0 0 -1 0 0 0 0 1 0 0 0;0 0 0 1 0 0 0 0 0 -1 0 0;
69         0 0 0 0 -1 0 0 0 0 1 0 0;0 0 0 0 1 0 0 0 0 0 -1 0]);

```

Listing D.4: PolyOp.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7
8 include("VariDeclared.jl") # Uses: PolyCone
9
10 # All Times functions correspond to lemma 4.2.5 in some way.
11
12 function Times(A::Vector{fmpz_mat},B::Vector{fmpz_mat}) # Neither A1 or A2 should be empty.
13     Amat=A[1]
14     Bmat=B[1]
15
16     for i in 2:length(A)
17         Amat=vcat(Amat,A[i])
18     end
19
20     for i in 2:length(B)
21         Bmat=vcat(Bmat,B[i])
22     end
23     A1=hcat(Amat,zero_matrix(ZZ,nrows(Amat),ncols(Bmat)))

```

```

24 B2=hcat(zero_matrix(ZZ,nrows(Bmat),ncols(Amat)),Bmat)
25 return vcat(A1,B1)
26
27 end
28
29 function Times(A::fmpz_mat,B::fmpz_mat)
30 A1=hcat(A,zero_matrix(ZZ,nrows(A),ncols(B)))
31 B1=hcat(zero_matrix(ZZ,nrows(B),ncols(A)),B)
32 return vcat(A1,B1)
33 end
34
35 function RowsToArr(V) # Takes Rows of a matrix to an array.
36 Mat=[]::Vector{fmpq_mat}
37
38 for i in 1:nrows(V)
39 Mat=vcat(Mat,V[i,:])
40 end
41 return Mat
42
43 end

```

Listing D.5: CalcInter.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7 import Base.convert
8
9 include("VariDeclared.jl") # Uses: PolyConeInfo.
10 include("PolyOp.jl")
11
12 # #-#-# # We follow section 4.4 closely. We call UpdatePolyI and InterInfo from other files.
13
14 function edgesINhalfspace(E,a) # Check if each edge has  $k \cdot a \geq 0$ .
15 for i in 1:length(E)
16 if dot(E[i],a)<0
17 return false
18 end
19 end
20 return true
21
22 end
23
24 function edgesINhyperplane(E,a) # Check that no Edge has  $k \cdot a > 0$ .
25 for i in 1:length(E)
26 if dot(E[i],a)>0
27 return false
28 end
29 end
30 return true
31
32 end
33
34 function GetEdges(E,a) # Get edges such that  $k \cdot a = 0$ .
35 Enew=fmpz_mat[]::Vector{fmpq_mat}
36

```

```

37   for i in 1:length(E)
38       if dot(E[i],a)==0
39           Enew=vcat(Enew,[E[i]])
40       end
41   end
42   return Enew
43
44 end
45
46 function sumEdges(E)
47     sum=0
48
49     for i in 1:length(E)
50         sum+=E[i]
51     end
52     return sum
53
54 end
55
56 function CheckIfEmpty(E,B) ::Bool # Using the criterion from section 4.4.
57     sum=sumEdges(E)
58
59     for i in 1:nrows(B)
60         if dot(sum,B[i,:])<=0
61             return true
62         end
63     end
64     return false
65
66 end
67
68 # #-#-# #
69
70 function VecToMat(E) # Array to matrix.
71     Enew=E[1]
72
73     for i in 2:length(E)
74         Enew=hcat(Enew,E[i])
75     end
76     return Enew
77
78 end
79
80 function NumberOfLinob(E) # Calculate the dimension.
81     return rank(VecToMat(E))
82 end
83
84 function GetNonRedundant(A,E,dim) # Checking for non-redunant constraints as in section 4.4.
85     Anew=A[1:4,:] # We want to keep B in Anew.
86
87     for i in 5:nrows(A) # Start from 5 to keep B in Anew.
88         count=0
89
90         for j in 1:length(E)
91             if dot(E[j],A[i,:])==0
92                 count+=1
93             end
94         end

```

```

95
96     if count >= dim-1
97         Anew = vcat(Anew, A[i,:])
98     end
99 end
100 return Anew
101
102 end
103
104 # #-#-# #
105
106 function GetZeroRows(A::fmpz_mat, e::fmpq_mat) # Get the rows a of A for which a \cdot e = 0.
107     Anew = "empty"
108
109     for i in 1:length(A[:,1])
110         if dot(A[i,:], e) == 0
111             if Anew == "empty"
112                 Anew = A[i,:]
113             else
114                 Anew = vcat(Anew, A[i,:])
115             end
116         end
117     end
118     return Anew
119
120 end
121
122 function GetZeroRows(A::fmpz_mat, E::Vector{fmpq_mat})
123     Ezero = fmpz_mat[] :: Vector{fmpz_mat}
124
125     for i in 1:length(E)
126         Ezero = vcat(Ezero, [GetZeroRows(A, E[i])])
127     end
128     return Ezero
129
130 end
131
132 function SetInter(A, B) # Getting the intersection for corollary B.0.5.
133     if A == "empty" || B == "empty"
134         return "empty"
135     else
136         C = "empty" # We start by assuming that C, which we will return, is empty.
137
138         for i in 1:nrows(A)
139             t = false
140             k = nrows(B)
141             j = 1
142
143             while t == false && j <= k
144                 if A[i,:] == B[j,:]
145                     t = true
146                 end
147                 j += 1
148             end
149
150             if t == true
151                 if C == "empty"
152                     C = A[i,:]

```

```

153         else
154             C=vcat(C,A[i,:])
155         end
156     end
157 end
158 end
159 return C
160
161 end
162
163 function Check2dim(S ::String)
164     return false
165 end
166
167 function Check2dim(A ::fmpz_mat) # Check dimension of the image.
168     if ncols(A)-rank(A)==2
169         return true
170     else
171         return false
172     end
173 end
174
175 function GetDots(E,a) # Pre-calculate the inner product of edges and a.
176     Edot=fmpz[] ::Vector{fmpz}
177
178     for i in 1:length(E)
179         Edot=vcat(Edot,dot(E[i],a))
180     end
181     return Edot
182
183 end
184
185 function Get2sides(A,E,a) # We determined 2-faces for lemma B.0.7.
186     twoSides=fmpz_mat[] ::Vector{fmpz_mat}
187     Edot=GetDots(E,a)
188     Ezero=GetZeroRows(A,E) # Vector of length length(E), keeping track of zero rows.
189
190     for i in 1:length(E)
191         for j in i+1:length(E)
192             if Edot[i]>0 && Edot[j]<0
193                 Ainter=SetInter(Ezero[i],Ezero[j])
194
195                 if Check2dim(Ainter) # According to corollary B.0.5.
196                     twoSides=vcat(twoSides,[hcat(E[i],E[j])])
197                 end
198             elseif Edot[i]<0 && Edot[j]>0
199                 Ainter=SetInter(Ezero[i],Ezero[j])
200
201                 if Check2dim(Ainter)
202                     twoSides=vcat(twoSides,[hcat(E[j],E[i])])
203                 end
204             end
205         end
206     end
207     return twoSides
208
209 end
210

```



```

211 function GetEgeq(E,a) # Get edges such that  $k \cdot a \geq 0$ .
212     Enew=fmpq_mat [] :: Vector{fmpq_mat}
213
214     for i in 1:length(E)
215         if dot(E[i],a)>=0
216             Enew=vcat(Enew,[E[i]])
217         end
218     end
219     return Enew
220
221 end
222
223 function DivMat(M::fmpq_mat,c::fmpq) # Divides matrix by a constant.
224     Mnew=zero_matrix(QQ,nrows(M),ncols(M))
225
226     for i in 1:nrows(M)
227         for j in 1:ncols(M)
228             Mnew[i,j]=M[i,j]//c
229         end
230     end
231     return Mnew
232
233 end
234
235 function GetInt2SHP(twoSides,a) # Calculates the new edges according to lemma B.0.7.
236     Enew=fmpq_mat [] :: Vector{fmpq_mat}
237
238     for i in 1:length(twoSides)
239         E1=twoSides[i][:,1]
240         E2=twoSides[i][:,2]
241         Enew=vcat(Enew,[DivMat(E1,abs(dot(E1,a)))+DivMat(E2,abs(dot(E2,a)))] #  $F \cap b^{\text{bot}}$ .
242     end
243     return Enew
244
245 end
246
247 # #-#-# #
248
249 function UpdatePolyI(PolyI ::PolyConeInfo,a ::fmpz_mat) # We assume PolyI is non-empty.
250     E=PolyI.E
251     A=PolyI.P.A
252     B=PolyI.P.B
253
254     if edgesINhalfspace(E,a) # Case 1 as in section 4.4.
255         return PolyI
256     elseif edgesINhyperplane(E,a) # Case 2 as in section 4.4.
257         Enew=GetEdges(E,a)
258         if isempty(Enew) || CheckIfEmpty(Enew,B)
259             return "empty"
260         else
261             dimnew=NumberOfLinob(Enew)
262             Anew=GetNonRedundant(vcat(A,a),Enew,dimnew)
263
264             return PolyConeInfo(PolyCone(Anew,B),Enew,dimnew)
265         end
266     else # Case 3 as in section 4.4.
267         twoSides=Get2sides(A,E,a)
268         Enew=vcat(GetEgeq(E,a),GetInt2SHP(twoSides,a))

```

```

269     Anew=GetNonRedundant(vcat(A,a),Enew,PolyI.dim)
270
271     return PolyConeInfo(PolyCone(Anew,B),Enew,PolyI.dim)
272 end
273 end
274
275 function UpdatePolyI(PolyI ::String,a ::fmpz_mat)
276     return "empty"
277 end
278
279 # #-#-# #
280
281 function CheckPolyIncluded(P ::PolyConeInfo,A ::fmpz_mat)
282     for i in 1:length(P.E)
283         for j in 1:nrows(A)
284             if dot(P.E[i],A[j,:])<0
285                 return false
286             end
287         end
288     end
289     return true
290
291 end
292
293 function InterInfo(P ::PolyConeInfo,A ::fmpz_mat)
294     # Updating PolyConeInfo with new constraints in the form of rows of a matrix.
295     # A kind of intersection => InterInfo.
296     for i in 1:nrows(A)
297         P=UpdatePolyI(P,A[i,:])
298     end
299     return P
300
301 end
302
303 function InterInfo(P ::PolyConeInfo,A ::Vector{fmpz_mat}) # Array instead of a matrix.
304     for i in 1:length(A)
305         P=UpdatePolyI(P,A[i])
306     end
307     return P
308
309 end
310
311 function InterInfo(P ::String, A)
312     return "empty"
313 end

```

Listing D.6: CalcMIN.jl

```

1  using Hecke
2  using Base: IO
3  import Base: show
4  import Base: ==
5  import Base: <=
6  import Base.gcd
7
8  function normF(x)
9      return x[1]^2+x[2]^2+x[3]^2
10 end
11

```

```

12 function CheckPreq(x::fmpz_mat,y::fmpz_mat)
13 # Check if x\preceq y according to lemma 5.4.1.
14 t=false
15
16 if abs(x[3])<= abs(y[3])
17 if y[3]^2+y[2]^2-x[3]^2-x[2]^2-abs(y[2]y[3]-x[2]x[3])>=0
18 a=normF(y)-normF(x)
19 b=min(0,y[1]y[2]-x[1]x[2])+min(0,y[1]y[3]-x[1]x[3])-abs(y[2]y[3]-x[2]x[3])
20 c=max(0,min(-y[1]y[2]+x[1]x[2],-y[1]y[3]+x[1]x[3],y[2]y[3]-x[2]x[3]))
21
22 if a+b+c>=0
23 t=true
24 end
25 end
26 end
27 return t
28
29 end
30
31 # #-#-# #
32
33 function MINfin(X) # Calculating MIN for a finite set in  $\mathbb{Z}^3$ .
34 minX=missing # minX can't be empty. For now it is missing.
35 s=false
36 k=length(X[1,:])
37
38 for i in 1:k # Go through each column of X to check if minimal.
39 t=true
40 j=1
41
42 while t==true && j<=k
43 if i!=j && normF(X[:,j])<=normF(X[:,i]) && CheckPreq(X[:,j],X[:,i])
44 # By proposition 5.1.9, we check the norms first.
45 t=false
46 end
47 j+=1
48 end
49 if t==true && s==false
50 minX=X[:,i]
51 s=true
52
53 elseif t==true
54 minX=hcat(minX,X[:,i])
55 end
56 end
57 return minX
58
59 end
60
61 # #-#-# #
62
63 function CheckZstar(v) # We check if v is in  $\mathbb{Z}^3$ .
64 t=false
65
66 if gcd(gcd(v[1],v[2]),v[3])==1
67 if v[3]>0
68 t=true
69 elseif v[3]==0 && v[2]>0

```

```

70     t=true
71     elseif v[3]==0 && v[2]==0 && v[1]>0
72         t=true
73     end
74 end
75 return t
76
77 end
78
79 # #-#-# #
80
81 function Wa(k,a) # The sets defined in section 5.4. Here, k corresponds to \Lambda.
82     if k==0
83         W=matrix(ZZ,[a 1 0])'
84         lim=floor(sqrt(2(a^2+max(a,0)+1)))
85     elseif k==1
86         W=matrix(ZZ,[a 0 1])'
87         lim=floor(sqrt(2(a^2+max(a,0)+1)))
88     else
89         W=matrix(ZZ,[a -1 1;a 1 1])'
90         lim=floor(sqrt(2(a^2+abs(a)+max(a,0)+3)))
91     end
92
93     for i1 in -lim:lim
94         for i2 in -lim:lim
95             for i3 in -lim:lim
96                 u=matrix(ZZ,[i1 i2 i3])'
97                 # We go through all possible finite vectors as noted above lemma 5.4.4.
98
99                 if k==0
100                     if CheckZstar(u)==true && CheckPreq(W[:,1],u)==false
101                         W=hcat(W,u)
102                     end
103                 elseif k==1
104                     if CheckZstar(u)==true && CheckPreq(W[:,1],u)==false
105                         if u[3]!=0
106                             W=hcat(W,u)
107                         end
108                     end
109                 else
110                     if u[2]!=0 && u[3]!=0 && CheckZstar(u)==true
111                         if CheckPreq(W[:,1],u)==false && CheckPreq(W[:,2],u)==false
112                             W=hcat(W,u)
113                         end
114                     end
115                 end
116             end
117         end
118     end
119     return W
120
121 end
122
123 # #-#-# #
124
125 function xInX(x,X) # Return true if x lies in X, otherwise return false.
126     t=false
127     i=1

```

```

128 k=length(X[1,:])
129
130 while t==false && i<=k
131     if x==X[:,i]
132         t=true
133     end
134     i+=1
135 end
136 return t
137
138 end
139
140 # #-#-# #
141
142 function GetSet(Y,X) # Find elements of Y that are not in X.
143     Ynew=false
144     s=false
145
146     for i in 1:length(Y[1,:])
147         if xInX(Y[:,i],X)==false
148             if s==false
149                 Ynew=Y[:,i]
150                 s=true
151             else
152                 Ynew=hcat(Ynew,Y[:,i])
153             end
154         end
155     end
156     return Ynew
157
158 end
159
160 # #-#-# #
161
162 function GetPosa(a) # Get position for the sets defined next.
163     if a>0
164         return 2*a
165     else
166         return 1-2*a
167     end
168 end
169
170 # #-#-# #
171
172 W_0=fmpz_mat[Wa(0,0)] :: Vector{fmpz_mat};
173 W_1=fmpz_mat[Wa(1,0)] :: Vector{fmpz_mat};
174 W_2=fmpz_mat[Wa(2,0)] :: Vector{fmpz_mat};
175
176 for i in 1:5 # Precalculation of Wa(k,a) sets.
177     global W_0=hcat(W_0,Wa(0,i),Wa(0,-i))
178     global W_1=hcat(W_1,Wa(1,i),Wa(1,-i))
179     global W_2=hcat(W_2,Wa(2,i),Wa(2,-i))
180 end
181
182 # #-#-# #
183
184 function MIN(Lam::fmpz,X::fmpz_mat) # We calculate MIN as in corollary 5.4.5.
185     Xout=missing

```

```

186 # Xout is the set we will return. By proposition 5.1.9,
187 # it is non-empty and for now we call it missing.
188
189 if X==zero_matrix(ZZ,1,1) # We write X==[0] if X is empty as a convention.
190   if Lam==0
191     Xout=MINfin(W_0[1])
192   elseif Lam==1
193     Xout=MINfin(W_1[1])
194   else
195     Xout=MINfin(W_2[1])
196   end
197 else
198   a=0
199   t=false
200
201   if Lam==0 # Proceed as in corollary 5.4.5 a).
202     while t==false
203       if xInX(matrix(ZZ,[-a 1 0])',X)==false
204         t=true
205
206         if abs(a)<=5
207           Xout=MINfin(GetSet(W_0[GetPosa(-a)],X))
208         else
209           Xout=MINfin(GetSet(Wa(Lam,-a),X))
210         end
211
212       elseif xInX(matrix(ZZ,[a 1 0])',X)==false
213         t=true
214
215         if abs(a)<=5
216           Xout=MINfin(GetSet(W_0[GetPosa(a)],X))
217         else
218           Xout=MINfin(GetSet(Wa(Lam,a),X))
219         end
220       end
221       a+=1
222     end
223
224   elseif Lam==1 # Proceed as in corollary 5.4.5 b).
225     while t==false
226       if xInX(matrix(ZZ,[-a 0 1])',X)==false
227         t=true
228
229         if abs(a)<=5
230           Xout=MINfin(GetSet(W_1[GetPosa(-a)],X))
231         else
232           Xout=MINfin(GetSet(Wa(Lam,a),X))
233         end
234
235       elseif xInX(matrix(ZZ,[a 0 1])',X)==false
236         t=true
237
238         if abs(a)<=5
239           Xout=MINfin(GetSet(W_1[GetPosa(a)],X))
240         else
241           Xout=MINfin(GetSet(Wa(Lam,a),X))
242         end
243       end

```

```

244     a+=1
245     end
246
247     elseif Lam==2 # Proceed as in corollary 5.4.5 c).
248     while t==false
249         if xInX(matrix(ZZ,[-a -1 1]'),X)==false && xInX(matrix(ZZ,[-a 1 1]'),X)==false
250             t=true
251
252             if abs(a)<=5
253                 Xout=MINfin(GetSet(W_2[GetPosa(-a)],X))
254             else
255                 Xout=MINfin(GetSet(Wa(Lam,-a),X))
256             end
257
258             elseif xInX(matrix(ZZ,[a -1 1]'),X)==false && xInX(matrix(ZZ,[a 1 1]'),X)==false
259                 t=true
260
261                 if abs(a)<=5
262                     Xout=MINfin(GetSet(W_2[GetPosa(a)],X))
263                 else
264                     Xout=MINfin(GetSet(Wa(Lam,-a),X))
265                 end
266             end
267             a+=1
268         end
269     end
270 end
271 return Xout
272
273 end
274
275 function MIN(Lam::fmpz,X::Vector{fmpz_mat})
276     if length(X)==0
277         return MIN(Lam,zero_matrix(ZZ,1,1))
278     end
279     Xmat=X[1]
280
281     for i in 2:length(X)
282         Xmat=hcat(Xmat,X[i])
283     end
284     return MIN(Lam,Xmat)
285
286 end

```

Listing D.7: CalcK.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7
8 include("CalcMIN.jl")
9
10 function K(Lam::fmpz,X::fmpz_mat) # As in lemma 5.1.11.
11     Kvec=fmpz_mat[] ::Vector{fmpz_mat}
12     k=length(X[1,:])
13

```

```

14 for i in 1:k-1
15     At1=matrix(ZZ,[X[1,i+1]^2-X[1,i]^2 X[2,i+1]^2-X[2,i]^2])
16     At2=matrix(ZZ,[X[3,i+1]^2-X[3,i]^2 2X[1,i+1]X[2,i+1]-2X[1,i]X[2,i]])
17     At3=matrix(ZZ,[2X[1,i+1]X[3,i+1]-2X[1,i]X[3,i] 2X[2,i+1]X[3,i+1]-2X[2,i]X[3,i]])
18     At=hcat(At1,At2,At3)
19
20     Kvec=vcat(Kvec,At)
21 end
22 Y=MIN(Lam,X)
23
24 for j in 1:length(Y[1,:])
25     At1=matrix(ZZ,[Y[1,j]^2-X[1,k]^2 Y[2,j]^2-X[2,k]^2])
26     At2=matrix(ZZ,[Y[3,j]^2-X[3,k]^2 2Y[1,j]Y[2,j]-2X[1,k]X[2,k]])
27     At3=matrix(ZZ,[2Y[1,j]Y[3,j]-2X[1,k]X[3,k] 2Y[2,j]Y[3,j]-2X[2,k]X[3,k]])
28     At=hcat(At1,At2,At3)
29
30     Kvec=vcat(Kvec,At)
31 end
32 return Kvec
33
34 end
35
36 function K(Lam::fmpz,X::Vector{fmpz_mat})
37     Xmat=X[1]
38
39     for i in 2:length(X)
40         Xmat=hcat(Xmat,X[i])
41     end
42     return K(Lam,Xmat)
43
44 end
45
46 # #-#-# #
47
48 function Tequal(x,y) # As seen in the definition of T_{xy}.
49     At1=matrix(ZZ,[x[1]^2 x[2]^2 x[3]^2 2x[1]x[2] 2x[1]x[3] 2x[2]x[3]])
50     At2=matrix(ZZ,[y[1]^2 y[2]^2 y[3]^2 2y[1]y[2] 2y[1]y[3] 2y[2]y[3]])
51     At=hcat(-At1,At2)
52
53     return vcat(At,-At)
54 end

```

Listing D.8: CalcVcrossV.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==
5 import Base: <=
6 import Base.gcd
7
8 include("VariDeclared.jl") # Uses: PolyConeInfo.
9 include("CalcInter.jl") # Uses: CheckPolyIncluded, UpdatePolyI.
10 include("PolyOp.jl")
11
12 CbLeft=fmpz_mat[] ::Vector{fmpz_mat}
13 CbRight=fmpz_mat[] ::Vector{fmpz_mat}
14 DbLeft=fmpz_mat[] ::Vector{fmpz_mat}
15 DbRight=fmpz_mat[] ::Vector{fmpz_mat}

```



```

16 Zero1_6=zero_matrix(ZZ,1,6)
17 Zero2_6=zero_matrix(ZZ,2,6)
18
19 for i in 1:9
20 # We define sets for lemma 5.2.1 that correspond to
21 #  $(0,c)^{\{\bot\}}, (c,0)^{\{\bot\}}, (0,d)^{\{\geq\}}, (d,0)^{\{\geq\}}$ .
22 global CbLeft=vcat(CbLeft,[hcat(vcat(-Cbound[i,:],Cbound[i,:]),Zero2_6)]);
23 global CbRight=vcat(CbRight,[hcat(Zero2_6,vcat(-Cbound[i,:],Cbound[i,:]))]);
24 global DbLeft=vcat(DbLeft,[hcat(Dbound[i,:],Zero1_6)]);
25 global DbRight=vcat(DbRight,[hcat(Zero1_6,Dbound[i,:]))];
26 end
27
28 # #-#-# #
29
30 function VcrossV(PolyI ::PolyConeInfo)
31 # As in lemma 5.2.1. Note that we can skip the first step, since it is redundant.
32 Ti=PolyI
33 t=false
34 bool1=zero_matrix(ZZ,1,9)
35 bool2=zero_matrix(ZZ,1,9)
36
37 while t==false
38 T=Ti
39 updatetrue=false # Used to check when no new updates happen
40
41 for i in 1:9 # 9 is the number of rows of Cbound
42 if bool1[1,i]==0 && CheckPolyIncluded(T,CbLeft[i])
43 bool1[1,i]=1
44 updatetrue=true
45
46 Ti=UpdatePolyI(Ti,DbLeft[i])
47 end
48
49 if bool2[1,i]==0 && CheckPolyIncluded(T,CbRight[i])
50 bool2[1,i]=1
51 updatetrue=true
52
53 Ti=UpdatePolyI(Ti,DbRight[i])
54 end
55 end
56
57 if updatetrue==false || Ti=="empty"
58 t=true
59 end
60 end
61 return Ti
62
63 end
64
65 function VcrossV(S::String)
66 return "empty"
67 end

```

Listing D.9: CalcLamNxy.jl

```

1 using Hecke
2 using Base: IO
3 import Base: show
4 import Base: ==

```

```

5 import Base: <=
6 import Base.gcd
7
8 # #-#-# #
9
10 function FindMaxR(Lam::fmpz,X::fmpz_mat,Y::fmpz_mat)
11 # If we get to this point, then we know that Lam is either 1 or 2.
12 s=false
13 c=length(X[1,:])
14 XBool=zeros_matrix(ZZ,1,length(X[1,:]))
15 YBool=zeros_matrix(ZZ,1,length(Y[1,:]))
16 # Note that XBool and YBool are of the same size.
17
18 for i in 1:length(XBool)
19     xCheck=X[:,i]
20     yCheck=Y[:,i]
21
22     if Lam==2
23         if xCheck[2]!=0 && xCheck[3]!=0 # We check that x is in  $Z_*^3 \setminus \text{Lambda}_2$ .
24             XBool[1,i]=1
25         end
26         if yCheck[2]!=0 && yCheck[3]!=0
27             YBool[1,i]=1
28         end
29     elseif Lam==1
30         if xCheck[3]!=0 # We check that x is in  $Z_*^3 \setminus \text{Lambda}_1$ .
31             XBool[1,i]=1
32         end
33         if yCheck[3]!=0
34             YBool[1,i]=1
35         end
36     end
37 end
38 while s==false && c>0
39     if sum(XBool[1,1:c])==sum(YBool[1,1:c])
40         # Find the biggest value c such that the sums are equal.
41         s=true
42     else
43         c-=1
44     end
45 end
46 return c
47
48 end
49
50 function FindMaxR(Lam::fmpz,X::Vector{fmpz_mat},Y::Vector{fmpz_mat})
51 # X and Y are always non-empty here.
52 Xmat=X[1]
53 Ymat=Y[1]
54
55 for i in 2:length(X)
56     Xmat=hcat(Xmat,X[i])
57     Ymat=hcat(Ymat,Y[i])
58 end
59 return FindMaxR(Lam,Xmat,Ymat)
60
61 end

```