On the Reduction of Quantum Teams

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Abstract

In this thesis, a reduction procedure on quantum teams is defined. Physically, this reduction procedure can be seen as an attempt to describe the measurement results of a certain (quantum mechanics) experiment purely classically, i.e. with hidden variables. In complete agreement with the expectation, the failure of this attempt indicates that genuine quantum effects are in play; the reduction procedure halts without having converted the team to a multi-team. It can therefore be used to demonstrate contextuality in a given quantum team. The reduction procedure conserves the corresponding probability table as well as all the properties expressible in Quantum Team Logic. Finally, an attempt has been made to solve the open problem of the axiomatisation of QTL formulas that agree with quantum mechanics. Absence of references to literature indicates original work from the author.

Contents

1 Introduction

In this thesis logic is applied to quantum mechanics. More specifically, logical machinery is applied to entanglement experiments in an attempt to give accurate and useful descriptions of the phenomena. Entanglement sparks the imagination of many; arguably because of its connections with teleportation. The phenomenon was first written about in [EPR1935] after which the literature of entanglement and, more generally, contextuality has expanded. The most notable contributions to this subject are [B1964], in which John S. Bell proved quantum mechanics to violate certain inequalities, and the contributions [CHSH1969] stating the famous CHSH inequalities. In both papers, we imagine a scenario like in Figure 1.1.

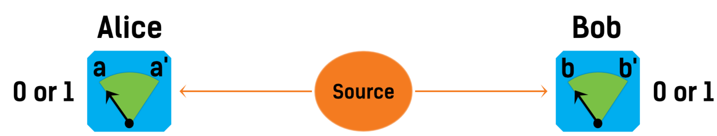

Figure 1.1: *Bell's scenario of type (2,2,2). Both Alice and Bob have measuring devices with two different settings. They measure particle pairs sent from a shared source, with which measurement outcomes are created, taking on values in* $\{0, 1\}$.

In the figure, Alice and Bob are two physicists who are measuring electron pairs emitted from the source. In this situation, the relevant property of electrons is their spin, which can be thought of as an actual spinning motion around their axis, creating a magnetic field due to the electric charge. This spin is quantised: it does not take on continuous values but instead takes on values in the discrete spectrum $\{0, 1\}$. The measurement that is being performed is a simple one conceptually. Incoming electrons pass through a magnetic field, which alters the electrons' paths. For a vertically oriented magnetic field the paths are either bent upwards or downwards, hence the dichotomous measurement values. The magnetic field can, in principle, have any orientation in space, but for the present, we assume it has two orientations only; these are called the measurement settings. Alice's measurement settings are labelled a and a' while Bob's are labelled b and b'.

Quantum mechanics allows the electron pairs to be entangled, which will cause certain correlations in Alice and Bob's measurements that cannot be explained by classical theories of physics, where all properties of a physical state are determined. The correlations violate the previously mentioned Bell inequalities, which would have to be satisfied in classical physics. A general perspective can be found by studying these inequalities from a logical point of view. In [AH2012] it is proved that all Bell inequalities can be derived from these logical Bell inequalities. In [HPV2016] a logic is designed that proves the violations of logical Bell inequalities.

A tabulation of the Measurement Results

In the experimental setup above, we can imagine Alice and Bob having registered their measurement outcomes as in table 1.

For the first three measurements Alice chose setting a and for the fourth she chose a', while Bob is alternating between b and b'. Both Alice and Bob can only choose one measurement setting, hence the blanks in the table, indicating the measurements that were not performed.

	a	a'	b	b'
0	1	-	0	-
1	1	-	-	1
2	0	-	1	-
3	-	0	-	0
...				

Table 1: *Alice's and Bob's measurements.*

One can imagine many different scenarios in which the measurement values are not dichotomous, but trichotomous or even continuous. Also, one can imagine many more different measurement settings, as well as more than two measuring agents. Later, we give a formal specification of all these scenarios.

Dependence logic and team semantics

The previously mentioned correlations between the measurement results from Alice and Bob indicate a certain dependence on their respective measurement outcomes. Dependence logic might therefore be a valuable tool to study these correlations from a logical perspective. [V2007] Forms a good reference. In the present context, what we need to understand from dependence logic, is its team semantics. An example of a team is table 2. It shows the colours of the shirts of players in a soccer team. The players in the field are wearing red while those on the bench are wearing white. There is one keeper with a black shirt.

The dependence logic formula $=(\text{Red}, \text{White}, \text{Black})$ informally means that the value of Black depends on the values of both Red and White. This means that for any two different rows s_i and s_j such that both $s_i(\text{Red}) = s_j(\text{Red})$ and $s_i(\text{White}) = s_j(\text{White})$ we have: $s_i(\text{Black}) = s_j(\text{Black})$. Dependence logic is not used in this thesis, so the team semantics just functions as an illustration. Looking forward, note that table 1 can also be seen as a team, although some of its fields are undetermined. A team where some of the values are undetermined is called a *quantum team*; a more formal definition follows later.

	Red	White	Black
s_0	0	1	0
s_1	1	0	0
s_2	0	1	0
s_3	1	0	0
s_4	0	1	0
s_5	0	0	1
s_6	1	0	0

Table 2: *A team of soccer players.*

Multi-teams

Observing Alice's and Bob's table, we can imagine the empty fields to contain definite values. This physically corresponds to a situation where the electron travelling to Alice, after being emitted from the source, contains information about the measurement results it will give for each of Alice's measurement settings. And similarly for the electron travelling towards Bob. In short, the unperformed measurements have values. This is an assumption from classical physics, because a classical physical state fully determines the results of any experiment that can be performed on it. The assumption is usually referred to by *hidden variables*. A quantum team containing all information on the unperformed measurements is called a *multi-team*, to be defined formally later. An example of a multi-team is shown in table 3. For the time being and for reasons that will become clear in the future, we call this multi-team (Ω, τ).

This thesis centres around a reduction procedure that aims to find a multi-team with the same explanatory power as a given quantum team. It is an attempt to describe the experimental results found by Alice and Bob in classical terms, i.e. as if there were hidden variables. As will be shown later, it can be used to demonstrate contextuality, a genuine quantum effect.

	p_0	p_1	p_2	p_3
0	1	1	0	1
1	1	1	0	1
2	1	1	1	1
3	1	1	1	0
4	0	0	1	1
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0

Table 3: *An example of a multi-team.*

Logical Bell inequalities

Multi-teams give rise to probabilities associated to propositions. For example, in table 3 we see that p_0 occurs with probability $[p_0]_{(\Omega,\tau)} = \frac{1}{2}$, while p_2 has an associated probability of $[p_2]_{(\Omega,\tau)} = \frac{3}{8}$.

In case of dichotomous measurement results, the measurement settings are boolean variables. This allows us to find probabilities associated to propositional formulas using a multi-team. Let $A = \{\phi_i\}$ be a set of formulas, each element of which has an associated probability $[\phi_i]_{(\Omega,\tau)}$. Let $k = |A|$ be the number of formulas. Using intuitive reasoning, we can understand the idea behind a logical Bell inequality:

$$\begin{aligned}
1 - \left[\bigwedge_i \phi_i\right]_{(\Omega,\tau)} &= \left[\neg \bigwedge_i \phi_i\right]_{(\Omega,\tau)} \\
&= \left[\bigvee_i \neg\phi_i\right]_{(\Omega,\tau)} \\
&\leq \sum_i [\neg\phi_i]_{(\Omega,\tau)} \\
&= \sum_i [1 - (\phi_i)]_{(\Omega,\tau)} \\
&= k - \sum_i [\phi_i]_{(\Omega,\tau)}
\end{aligned}$$

Which after reordering becomes:

$$\sum_i [\phi_i]_{(\Omega,\tau)} \leq k - 1 + \left[\bigwedge_i \phi_i\right]_{(\Omega,\tau)}$$

When A is contradictory, the associated probability to its conjunction is zero: $[\bigwedge_i \phi_i]_{(\Omega,\tau)} = 0$. In this case we get:

$$\sum_i [\phi_i]_{(\Omega,\tau)} \leq k - 1$$

This last expression is what we call a logical Bell inequality. When in the following reference is made to Bell inequalities, the logical type above is meant. Quantum mechanics predicts that these inequalities can be violated. If $[\phi_i]_{(\Omega,\tau)} = 1$ for all i, the inequality is maximally violated by 1.

Probabilistic Team Logic (PTL)

In [HPV2016] Probabilistic Team Logic (PTL) is introduced. Later in this thesis, we are going to use Quantum Team Logic (QTL) which closely resembles PTL. It is discussed for introductory purposes. PTL proves the logical Bell inequalities. The atomic formulas are the following:

Definition 1. Suppose $\phi_0, ..., \phi_k$ are propositional formulas, $(a_j)_{j \leq k} \in \mathbb{Z}$ and $c \in \mathbb{Z}$, then

$$a_0\phi_0 + ... + a_k\phi_k \geqslant c$$

is an atomic formula of PTL.

Definition 2. The set of formulas of PTL is defined as follows:

- Atomic formulas are formulas;
- If α is a formula, then $\neg\alpha$ is a formula;
- If α and β are formulas, then $\alpha \wedge \beta$ is a formula.

Definition 3. (Semantics). Suppose X is a multi-team and α a formula of PTL with all its propositional symbols corresponding to a column of X. We define by induction on α the relation $X \models \alpha$ in the following way:

- $X \models a_0\phi + ... + a_{k-1}\phi_{k-1} \geqslant c$ iff $a_0[\phi_0]_X + ... + a_{k-1}[\phi_{k-1}]_X \geqslant c$;
- $X \models \neg\alpha$ iff $X \not\models \alpha$;
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.

Here, $[\phi_i]_X$ is the probability associated to the propositional logic formula ϕ_i in the multi-team X. PTL proves the Bell inequalities, which makes it an unsuitable tool to adequately describe the experiment under discussion since these inequalities are violated. Later, we will be looking at Quantum Team Logic (QTL), which is a modification of PTL. QTL will be able to prove violations of Bell inequalities, which makes it better suitable to describe the experiment.

Notation

To formally describe the theory, we use the notation similar to [HPV2016], [AH2012] and [AB2011]. Firstly, we assume a finite set of agents A, a finite set of measurement settings M_i for $i \in A$, and a set of measurement values V_i^j for $j \in M_i$. The measurement settings are boolean variables in case $|V_i^j| = 2$; in other cases they are multi-valued propositional logic variables. To characterise a typical Bell scenario, we can write the finite sequence (n, k, l), meaning that $|A| = n$, $|M_i| = k$ for each i, and $|V_i^j| = l$ for each i and j. A Bell scenario is therefore one in which each agent has the same number of measurement settings to choose from, each of which can take on the same number of possible values. We say that such scenario is of *type* (n, k, l).

Picking one measurement setting from $M_i = \{p_i^m | m \in \{0, ..., k-1\}\}$ for each $i \in A$ gives us a *measurement context* $U = \{p_i^{m_i} | i \in A\}$. It represents an experimental setup where every agent has chosen one setting on his/her device.

The set $X = \cup_{i \in A} M_i$ is called the *measurement set*. It contains all measurement settings for every agent.

The *measurement cover* $\mathcal{U}$ on a measurement set X is the set of all measurement contexts. There are k^n many of them. Formally, we have $\mathcal{U} = \{U_j | j \in \{0, ..., k^n - 1\}\}$ where we assumed some numbering of the measurement contexts.

Each measurement can take on l values; $\mathbf{l} = \{0, ..., l-1\}$ denotes the set of these values. The set $\mathbf{l}^{U_i}$ of functions from a measurement context U_i to $\mathbf{l}$ is thought of as the *set of measurement outcomes*; each element of it is a *measurement outcome*. Note that these are thought of as measurement values of measurements that have actually been carried out. Let U be a measurement context and d_U a probability distribution on $\mathbf{l}^U$. The set $\{d_U\}_{U \in \mathcal{U}}$ is a *probability model* or *probability table* on the cover $\mathcal{U}$. The probability model gives the probabilities for each measurement outcome within each measurement context. The set $S(U) = \left\{s \in \mathbf{l}^U | d_U(s) > 0\right\}$ is the *support*; the collection of measurement outcomes that actually occur, i.e. have strictly positive probability. An assignment $\varsigma : X \to \mathbf{l}$ such that for all $U \in \mathcal{U}$ we have $\varsigma | U \in S(U)$ is called a *global section* for the support.

A *multi-team* is a pair (Ω, τ) where Ω is a non-empty set of *rows* and τ is a function on Ω such that for every $i \in \Omega$, $\tau(i)$ is an assignment for the measurement set X. The *size* of the multi-team Ω is the cardinality $|\Omega|$. In the following, we assume only finite multi-teams.

We denote the restriction $\tau|U$ of τ to a certain measurement context U by τ_U. The *associated probability table* to a multi-team (Ω, τ) is the set $\{d_U\}_{U \in \mathcal{U}}$ where

$$d_U(v) = \frac{|\{i \in \Omega | \tau(i) = v\}|}{|\Omega|}$$

with v an assignment for U.

In the case of dichotomous measurement values, the measurement settings are boolean, which gives us a notion of satisfaction. We can use this to extend probabilities to logical formulas ϕ, with propositional variables in U, as:

$$[\phi]_{(\Omega,\tau),U} = \sum_v d_U(v | v \text{ satisfies } \phi)$$

which we usually write as $[\phi]_U$ when the quantum team is known from the context.

Let $\mathcal{U}$ be non-empty and let Ω be an index set the elements of which we call *rows.* We assume a function $i \mapsto U_i$ from Ω to $\mathcal{U}$ and name this the *cover function.* This function associates the relevant measurement context to each element in Ω, i.e. the set of propositional variables that have an assignment in that row. We call U_i the *associated measurement context to i.* A *quantum team* on $\mathcal{U}$ is a pair (Ω, τ) such that Ω is non-empty and $\tau(i)$ is a truth value assignment to the proposition symbols in $U_i \in \mathcal{U}$ for all $i \in \Omega$. If all the U_i are the same, then our quantum team is in fact a multi-team. The number of different measurement contexts is denoted $N = |\mathcal{U}| = k^n$.

For a quantum team (Ω, τ) on $\mathcal{U} = \{U_i\}_{i \in \Omega}$, we write $\Omega_U = \{i \in \Omega | U \subseteq U_i\}$. The associated probability table for (Ω, τ) is the set $\{d_U\}_{U \in \mathcal{U}}$ where

$$d_U(v) = \frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|}$$

Which also extends to probabilities of logical formulas ϕ, with propositional variables in U, as:

$$[\phi]_{(\Omega,\tau),U} = \sum_v d_U(v | v \text{ satisfies } \phi)$$

which we usually write as $[\phi]_{(\Omega,\tau)}$ when the set of propositional variables in ϕ coincides with U.

For a summary of some part of the terminology, please observe the table depicted in figure 1.2. This table is referred to as Bell's table and is the associated probability table of the quantum team displayed in table 4 which is contextual, i.e. displays genuine quantum behaviour. The notion of contextuality comes in different grades and is defined in the next section.

	p_0	p_1	p_2	p_3
0	1	1	-	-
1	1	1	-	-
2	1	1	-	-
3	1	1	-	-
4	0	0	-	-
5	0	0	-	-
6	0	0	-	-
7	0	0	-	-
8	1	-	-	1
9	1	-	-	1
10	1	-	-	1
11	0	-	-	1
12	1	-	-	0
13	0	-	-	0
14	0	-	-	0
15	0	-	-	0
16	-	1	1	-
17	-	1	1	-
18	-	1	1	-
19	-	0	1	-
20	-	1	0	-
21	-	0	0	-
22	-	0	0	-
23	-	0	0	-
24	-	-	1	1
25	-	-	1	0
26	-	-	1	0
27	-	-	1	0
28	-	-	0	1
29	-	-	0	1
30	-	-	0	1
31	-	-	0	0

Table 4: *A Bell scenario of type* $(2,2,2)$. *This quantum team gives rise to Bell's table. It shows measurement outcomes that Alice and Bob could in principle have measured. Note that to reliably demonstrate violation of Bell's inequalities, many more measurements should be made to reach statistical significance. So this table serves for illustration purposes only.*

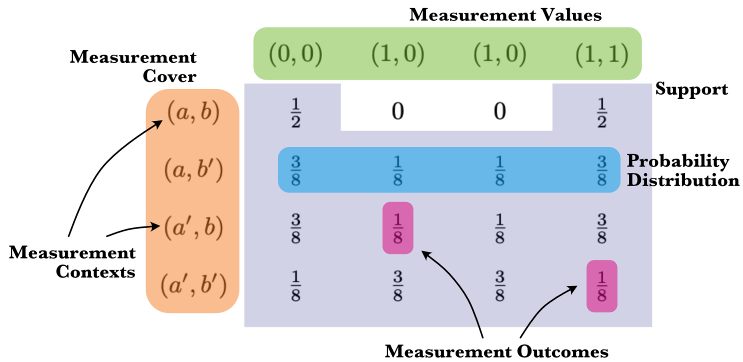

Figure 1.2: *The probability table that is referred to as Bell's table for a* $(2, 2, 2)$ *type scenario. Terminology indicated. For its derivation, see the appendix.*

Contextuality

Contextuality is the quantum phenomenon that refers to the fact that the measurement results of a quantum experiment depend on the measurement context (the measurement devices and their settings). In this thesis we restrict our attention to a specific instance of contextuality: violations of logical Bell inequalities. Contextuality then comes in a hierarchy of flavours, ordered by strength. The following result from [HPV2016] determines our way of presentation.

Lemma 4. *([HPV2016], 5.3) Every probability table with rational probabilities is the associated table of some quantum team.*

We will therefore mainly look at (rational) probability tables to illustrate the hierarchy. These tables take up less space on the page and bear a closer connection to the logical Bell inequalities because they readily show the probabilities.

Firstly, there is non-contextuality. This corresponds to classical physics where Bell inequalities are not violated. It is formally defined by:

Definition 5. A quantum team (Ω, τ) is *non-contextual* if all Bell inequalities $\sum_{i=0}^{k} [\phi_i]_{(\Omega,\tau)} \leq k-1$ with ϕ_i propositional formulas with variables in U_i, are satisfied.

For example, a quantum team with associated probability table 5 is non-contextual; any Bell inequality is satisfied.

Secondly, the weakest form of contextuality, which is called probabilistic contextuality. This corresponds to a genuine quantum situation that cannot be explained in classical terms using hidden variables.

Definition 6. A quantum team is *probabilistically contextual* if a Bell inequality is violated.

The quantum team in table 4 with associated table displayed in figure 1.2 is an example of a probabilistically contextual quantum team. Namely, the formulas $\phi_0 = a \leftrightarrow b$, $\phi_1 = a \leftrightarrow b'$, $\phi_2 =$

	$(0,0)$	$(0,1)$	$(1,0)$	$(0,0)$
(a,b)	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a,b')	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
(a',b)	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$
(a',b')	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{8}$

Table 5: *A probability table associated to a non-contextual quantum team in a* $(2,2,2)$ *type Bell scenario.*

$a' \leftrightarrow b$, and $\phi_3 = a' \oplus b' = \neg(a' \leftrightarrow b')$ are contradictory, but their total probability is $\sum_i [\phi_i]_{U_i} = 1 + \frac{6}{8} + \frac{6}{8} + \frac{6}{8} = 3\frac{1}{4} \geq 3$.

Thirdly, there is a stronger notion of contextuality called possibilistic contextuality. It implies probabilistic contextuality.

Definition 7. A quantum team is *possibilistically contextual* if there exists an element $s \in S(U)$ of its support such that there is no global section ς with $\varsigma|_U = s$.

The quantum team in table 4 is not possibilistically contextual, as for every element in its support a global section can be found. An example of possibilistic contextuality can be found in the quantum team associated to table 6. For the measurement outcome $s = \{a' \mapsto 1, b' \mapsto 0\}$ no global section ς exists such that $\varsigma|_{\{a',b'\}} = s$, as can be verified by inspection of the probability table.

Possibilistic contextuality implies probabilistic contextuality, as proved in [AH2012]:

Proposition 8. *([AH2012], III.1) Any possibilistically contextual model violates a logical Bell/CHSH inequality.*

Lastly, there is strong contextuality.

Definition 9. A quantum team is *strongly contextual* if there exists no global section.

A global section is an assignment ς such that $\varsigma|_U$ is in the support. This means that the formulas defining the support of a strongly contextual quantum team are not satisfiable.

The previous quantum team corresponding to 6 is not strongly contextual; the formulas defining its support are satisfiable. Table 7 shows what is known in the literature as the Popescu-Rohrlich box. Any corresponding quantum team is strongly contextual. The formulas defining its support are precisely the formulas ϕ_i defined in the definition of probabilistic contextuality which are contradictory.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a,b')	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a',b)	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a',b')	$\frac{1}{2}$	0	$\frac{1}{8}$	$\frac{3}{8}$

Table 6: *The associated probability table of a possibilistically contextual quantum team in a type* $(2,2,2)$ *Bell scenario. The measurement outcome* $\{a' \mapsto 1, b' \mapsto 0\}$ *has no corresponding global section.*

In [AH2012], the following result is proven:

Proposition 10. *([AH2012], III.2) A model achieves maximal violation of a logical Bell inequality if and only if it is strongly contextual.*

Strong contextuality therefore implies possibilistic contextuality as well as probabilistic contextuality.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a,b')	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a',b)	$\frac{1}{2}$	0	0	$\frac{1}{2}$
(a',b')	0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 7: *The Popescu-Rohrlich box with maximal violation. The Bell scenario is of type* $(2,2,2)$.

Please note that the Popescu-Rohrlich box displays a hypothetical situation. Quantum mechanics does not predict the violation of Bell inequalities to be this big. Incidentally, the situation still exemplifies non-signalling, so even strong contextuality does not imply the possibility of sending signals with arbitrary velocity.

2 Reduction

In this section we define our reduction procedure. We can *reduce* a quantum team (Ω, τ) to one of a simpler form in case we have the following:

- There is a subset $C \subset \Omega$, $|C| = N$ such that for $i, j \in \Omega$ if $i \neq j$ the associated measurement contexts are different $U_i \neq U_j$.
- There exists an assignment ς on X such that for all $i \in \Omega$, we have $\varsigma|_{U_i} = \tau(i)$.

Now we can produce a structure (Ω_1, τ_1) which is the same as (Ω, τ), except that all $i \in C$ are excluded and replaced by one element j with $U_j = X$. Now τ_1 on Ω_1 coincides with τ on $\Omega \cap \Omega_1$ and $\tau_1(j) := \varsigma$. The quantum team (Ω_1, τ_1) is *reduced* from the quantum team (Ω, τ).

This procedure of finding the next reduced quantum team can be repeated, but not indefinitely. When this process stops, let's say after m steps, we say that the resulting quantum team (Ω_m, τ_m) is in *reduced form.*

Lemma 11. *The reduced form is not unique in a Bell scenario of type* $(2, 2, 2)$.

Proof. Firstly, we consider two quantum teams the same if they merely differ in a permutation/relabelling of their rows and columns. Let's reduce table 4 in two different ways. First we replace rows 0, 8, 16 and 24 by a row assigning $p_0 = p_1 = p_2 = p_3 = 1$. Then we replace rows 1, 9, 20 and 28 by a row assigning $p_0 = p_1 = p_3 = 1$ and $p_2 = 0$. We replace rows 2, 12, 17 and 25 by a row assigning $p_0 = p_1 = p_2 = 1$ and $p_3 = 0$. We replace rows 4, 11, 21 and 29 by a row assigning $p_0 = p_1 = p_2 = 0$ and $p_3 = 1$. We replace the rows 5, 13, 22 and 31 by a row assigning $p_0 = p_1 = p_2 = p_3 = 0$. And lastly, we replace the rows 6, 14, 19 and 26 by a row assigning $p_0 = p_1 = p_3 = 0$ and $p_2 = 1$. The resulting quantum team cannot be reduced further. It is displayed on the left of Table 8.

Another quantum team can be obtained from the initial table by first replacing the rows 0, 12, 20 and 31 by a row that assigns $p_0 = p_1 = 1$ and $p_2 = p_3 = 0$. Then we replace rows 1, 8, 16 and 24 by a row assigning $p_0 = p_1 = p_2 = p_3 = 1$. Then we replace the rows 4, 13, 19 and 25 by a row that assigns $p_0 = p_1 = p_3 = 1$ and $p_2 = 0$. And lastly, we replace the rows 5, 11, 21 and 28 by a row assigning $p_0 = p_1 = p_2 = 0$ and $p_3 = 3$. This quantum team cannot be reduced further. It is displayed on the right of Table 8.

The two resulting tables are different; so, the reduced form of a quantum team is not unique. □

Remark. Even though the reduction procedure is not unique, in the following we will sometimes refer to it as *the* reduction procedure, with the understanding that multiple procedures are meant.

	p_0	p_1	p_2	p_3
0	1	1	1	1
1	1	1	0	0
2	1	1	1	0
3	0	0	0	1
4	0	0	0	0
5	0	0	1	0
6	1	1	-	-
7	0	0	-	-
8	1	-	-	1
9	0	-	-	0
10	-	1	1	-
11	-	0	0	-
12	-	-	1	0
13	-	-	0	1

	p_0	p_1	p_2	p_3
0	1	1	0	0
1	1	1	1	1
2	0	0	1	0
3	0	0	0	1
4	1	1	-	-
5	1	1	-	-
6	0	0	-	-
7	0	0	-	-
8	1	-	-	1
9	1	-	-	1
10	0	-	-	0
11	0	-	-	0
12	-	1	1	-
13	-	1	1	-
14	-	0	0	-
15	-	0	0	-
16	-	-	1	0
17	-	-	1	0
18	-	-	0	1
19	-	-	0	1

Table 8: *Two resulting Hybrid Teams.*

Definition 12. A quantum team (Ω, τ) is *hybrid* if $\Omega = \Omega_1 \cup \Omega_2$, where:

- $(\Omega_1, \tau|_{\Omega_1})$ is multi-team, i.e. $U_i = X$ for all $i \in \Omega_1$. We call this the *non-contextual part* of the hybrid team.
- $(\Omega_2, \tau|_{\Omega_2})$ is a quantum team which violates all Bell inequalities maximally. We call this the *contextual part* of the hybrid team.

When the contextual part is non-empty, we say that the hybrid team is *non-trivial*.

Lemma 13. *In a type $(n, k, 2)$ Bell scenario, a quantum team (Ω, τ) is of reduced form iff it is hybrid.*

Proof. Firstly, a quantum team (Ω, τ) of reduced form is a hybrid team. Due to the dichotomous measurement values, we can use propositional logic to describe the quantum team and to construct the proof. Let Ω' be the part of the quantum team with no fully determined rows. If this part is empty, we have a trivial hybrid team and we are done, so let's assume otherwise.

Let q_{ij} denote the propositional logic formula which only satisfying valuation corresponds to the ith row of Ω'_{U_j}. That is[1]:

$$q_{ij} = \bigwedge_{\tau|_{U_j}(i)(p)=1} p \wedge \bigwedge_{\tau|_{U_j}(i)(p)=0} \neg p$$

[1]For example, if $U_j = \{a, b'\}$ and $\tau(i)(a) = 0$ and $\tau(i)(b') = 1$, then $q_{ij} = \neg a \wedge b'$, such that the only valuation making q_{ij} true is precisely the valuation at the ith row of Ω'_{U_j}.

We denote the number of rows of Ω'_{U_j} by M_j. A valuation making:

$$q_{i_1 1} \wedge \ldots \wedge q_{i_N N}$$

true, is precisely a global section.[2]

By assumption, we cannot find an assignment ς such that $\varsigma|_{U_{i_j}} = \tau(i_j)$ for N different rows $i_0, ..., i_{N-1} \in \Omega$, since we are at the end of the reduction procedure. So all the conjunctions like the above are unsatisfiable. So the disjunction of all those conjunctions is unsatisfiable in this case. This big disjunction is equivalent – by distributing conjunctions over disjunctions – to a conjunction with terms of the form:

$$q_{1j} \vee \ldots \vee q_{M_1 j}$$

where j ranges over the measurement contexts $\{1, \ldots, N\}$. This conjunction precisely defines the support. So, when we cannot find a global section, the formula defining the support of the contextual part is not satisfiable, which means that the contextual part of the quantum team is *strongly contextual.* Applying proposition 10 now proves that the contextual part maximally violates the logical Bell inequality. The part $\Omega - \Omega'$ is a quantum team with only fully determined rows, therefore it is a multi-team. Concluding, a quantum team of reduced form is a hybrid team.

For the converse, we show that a hybrid team (Ω, τ) cannot be reduced. It is sufficient to check that the contextual part (Ω', τ') cannot be reduced, as the non-contextual part of the hybrid team, of course, does not even qualify for reduction. (Ω', τ') violates Bell's inequalities maximally. Let $\{d'_U\}_{U \in \mathcal{U}}$ be its probability table. Let p_v denote the propositional formula that is only satisfied for valuation $v \in \mathbf{1}^U$. We define $P_U = \{p_v | v \in \mathbf{1}^U\}$ to be the set of propositional formulas that define each possible valuation v on U. Let $P_U|+ = \{p_v \in P_U | d'_U(p_v \mapsto v) > 0\}$ be the subset of these formulas that have positive probability. P_U contains all propositional formulas p_v such that there is a $j \in \Omega'$ with $\tau'(j)(p_v) = 1$. Let $r_U = \bigvee_{p_v \in P_U|+} p_v$. r_U is a propositional formula defining the measurement outcomes with positive probability for a measurement context U. The formula:

$$\bigwedge_{U \in \mathcal{U}} r_U$$

therefore, defines the support of (Ω', τ'). This formula is contradictory by proposition 10 and our assumption on maximal violation.

On the other hand, finding an assignment ς such that $\varsigma|_{U_{i_j}} = \tau(i_j)$ for N different rows $i_0, \ldots, i_{N-1}$ is equivalent to finding $p^i_v \in U_i$ such that $\bigwedge_i p^i_v$ is satisfiable, where i indexes the measurement contexts. However, we see that:

$$p^i_v \to r_{U_i}$$

by disjunction introduction. Combining this with the conjuction above, we see that $\bigwedge_i p^i_v \to \bigwedge_i r_{U_i} \to \perp$, so such an assignment ς cannot be found. Impossibility of any reduction step now proves the lemma. □

[2] In the expression, for any j, i_j is in the range $\{1, \ldots, M_j\}$. So there is one row for each of the N measurement contexts. The conjunction shows that the assignment satisfies them all.

Lemma 14. *Any reduction procedure is probability conserving.*

Proof. We shall prove that the corresponding probability table of a quantum team does not change during one reduction step. This implies directly that it does not change over the whole reduction procedure, which consists of a finite number of steps.

Let (Ω, τ) be a quantum team with cover $\mathcal{U}$ of which we assume that it can be reduced. Let $\{d_U\}_{U \in \mathcal{U}}$ be its corresponding probability table. By definition,

$$d_U(v) = \frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|}$$

Now we perform a reduction step, which means that we can find a global section. So we choose N different $i_j \in \Omega$, $j \in \{0, \ldots, N-1\}$ such that there is a $\varsigma : X \to \mathbf{1}$ with $\varsigma|_{U_{i_j}} = \tau(i_j)$, where U_{i_j} denotes the measurement context pertaining to i_j. We replace all $i_j \in \Omega$ by this single ς and get the quantum team (Ω', τ'). The corresponding probability table is $\{d'_U\}_{U \in \mathcal{U}}$ where

$$d'_U(v) = \frac{|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{|\Omega'_U|}$$

Note now that $\Omega'_U = \{i \in \Omega' | U \subseteq U_i\}$. We are going to prove that for any U and any $i \in \Omega_U$, there is an $i' \in \Omega'_U$ and vice versa. There are two cases to consider. Firstly, $i \neq i_j$ for any j. In that case, $i = i'$, because the reduction step did not substitute this row, which means it is still there.

In the second case, $i = i_j$ for some j. This i now, corresponds to ς. The reason is that $U_\varsigma = X$, such that $U \subseteq U_\varsigma$ for any subset U of X. So $\varsigma \in \Omega'_U$ for any U.

So we proved that for every $i \in \Omega_U$, there is an $i' \in \Omega'_U$. It is easy to see that a similar reasoning applies when we want to prove the converse. Concluding, we have proved that $|\Omega_U| = |\Omega'_U|$. The following equality holds because $\tau_U = \tau'_U$.

$$\begin{aligned} d'_U(v) &= \frac{|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{|\Omega'_U|} \\ &= \frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|} \\ &= d_U(v) \end{aligned}$$

And we see that the corresponding probability table remains unchanged. □

3 Quantum Team Logic (QTL)

As we have seen, the logical Bell inequalities are proved by probability team logic, as well as propositional logic with measurement covers and probabilities associated to the propositions. We have also seen that these inequalities are an inadequate description of reality, because quantum mechanics predicts these inequalities to be violated in certain circumstances (see figure 1.2), a fact that is supported by experimental evidence. Considering this, we would like to use a different logic that is able to prove violations of the Bell inequalities. The logic we are using here is called Quantum Team Logic (QTL) and is developed in [HPV2016]. Its syntax and semantics are adopted from that paper; the rest of this section is original work.

Definition 15. We assume a sequence of propositional formulas $(\phi_i)_{i<k}$, a sequence of whole numbers $(a_i)_{i<k} \in \mathbb{Z}^k$, a number $c \in \mathbb{Z}$, and a sequence of finite sets of propositions $(U_i)_{i<k}$, such that the proposition symbols of ϕ_i are in U_i for every $i < k$. An *atomic formula of QTL* is of the form:

$$a_0\,(\phi_0; U_0) + \ldots + a_{k-1}\,(\phi_{k-1}; U_{k-1}) \geq c$$

These $(\phi_i; U_i)$ represents the proposition ϕ_i, together with the measurement context U_i on which it is defined. This is important, as it gives us a notion of satisfaction of a proposition without the requirement of using global sections. The numbers allow us to express the relevant (rational) probabilities.

Definition 16. The set of formulas $\mathcal{F}$ of QTL is defined inductively as:

- $\alpha \in \mathcal{F}$ for atomic formulas α
- $\alpha \in \mathcal{F}$ implies $\neg\alpha \in \mathcal{F}$
- $\alpha \in \mathcal{F}$ and $\beta \in \mathcal{F}$ together imply $\alpha \wedge \beta \in \mathcal{F}$
- $\mathcal{F}$ contains nothing else

The atomic formulas look like arithmetical expressions. As such, we will use abbreviations like the following:

- $(\phi; U_i) - (\psi; U_j) \geq c$ for $(\phi; U_i) + (-1)\,(\psi; U_j) \geq c$
- $(\phi \geq \psi; U_i)$ for $(\phi; U_i) - (\psi; U_i) \geq 0$
- $(\phi; U_i) = c$ for $((\phi; U_i) \geq c) \wedge ((\phi; U_i) \leq c)$

etc.

Let α be a formula of QTL. We define its *Context set*, $\mathrm{Cs}\,(\alpha)$ inductively as[3]:

[3] In [HPV2016] this notion is called the *Support*. Here, the term *Context set* is chosen instead, to avoid confusion with the other notion that was given the same name.

- $\mathrm{Cs}\left(\sum_{i<k} a_i\left(\phi_i; U_i\right) \geq c\right) = \{U_i | i < k\}$
- $\mathrm{Cs}(\neg\alpha) = \mathrm{Cs}(\alpha)$
- $\mathrm{Cs}(\alpha \wedge \beta) = \mathrm{Cs}(\alpha) \cup \mathrm{Cs}(\beta)$

We use the following notation in the definition of the semantics:

$$\mathcal{U} \leq_c \mathcal{U}' \Longleftrightarrow \forall U \in \mathcal{U} \exists U' \in \mathcal{U}' \left(U \subseteq U'\right)$$

It allows us to express an important condition stating that each measurement context in the measurement set is contained in some measurement set of the quantum team.

Definition 17. (Semantics) Let α be a QTL formula and (Ω, τ) a quantum team with $\mathrm{Cs}(\alpha) \leq_c \{U_i | i \in \Omega\}$. We define by induction on α the relation $(\Omega, \tau) \vDash \alpha$ by:

- $(\Omega, \tau) \vDash \sum_{i<k} a_i\left(\phi_i; U_i\right) \geq c$ iff $\sum_{i<k} a_i\left[\phi_i\right]_{(\Omega,\tau),U_i} \geq c$
- $(\Omega, \tau) \vDash \neg\alpha$ iff $(\Omega, \tau) \nvDash \neg\alpha$
- $(\Omega, \tau) \vDash \alpha \wedge \beta$ iff $(\Omega, \tau) \vDash \alpha$ and $(\Omega, \tau) \vDash \beta$

With both the syntax and semantics of QTL defined, we can use it to describe and compare quantum teams. Note that $[\phi_i]_{(\Omega,\tau),U_i}$ assumes the Bell scenario under discussion to be of type $(n, k, 2)$, because it uses the notion of satisfaction which relies on dichotomous measurement results.

A notion of equivalence in terms of QTL semantics is straightforwardly defined.

Definition 18. Quantum teams (Ω, τ) and (Ω', τ') are *equivalent* if they satisfy the same formulas of QTL, i.e. $(\Omega, \tau) \models \alpha$ iff $(\Omega', \tau') \models \alpha$ for any QTL formula α.

Note that the definition implies that the two quantum teams have the same cover, for if this were not the case, there would be a cover containing a measurement context U which is not contained in the other cover; in that case, the QTL formula $(\bigwedge_{p \in U} p; U) \geq 0$ is satisfied in one, but not in the other quantum team, so the quantum teams are not equivalent.

Lemma 19. *In (n,k,2) type Bell scenarios, quantum teams (Ω, τ) and (Ω', τ') are equivalent iff they have the same corresponding probability table.*

Proof. By induction on the complexity of QTL formulas, first the atomic case. Because of equivalence, for any atomic formula $\sum_{j<k} a_j\left(\phi_j; U_j\right) \geq c$ the following holds:

$$(\Omega, \tau) \models \sum_{j<k} a_j\left(\phi_j; U_j\right) \geq c \text{ iff } (\Omega', \tau') \models \sum_{j<k} a_j\left(\phi_j; U_j\right) \geq c$$

This means, by definition, that:

$$\sum_{j<k} a_j\left[\phi_j\right]_{(\Omega,\tau),U_j} \geq c \text{ iff } \sum_{j<k} a_j\left[\phi_j\right]_{(\Omega',\tau'),U_j} \geq c$$

Therefore, $[\phi_j]_{(\Omega,\tau),U_j} = [\phi_j]_{(\Omega',\tau'),U_j}$ for any ϕ_j (take $k = 1$) and we thus have, by definition, for any $U \in \mathcal{U}$:

$$P_{(\Omega,\tau),U}\left(\{i \in \Omega_U | \tau_U(i)(\phi) = 1\}\right) = P_{(\Omega',\tau'),U}\left(\{i \in \Omega'_U | \tau'_U(i)(\phi) = 1\}\right)$$

Now we can conclude that

$$\begin{aligned} d_U(v) &= \frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|} \\ &= \frac{|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{|\Omega'_U|} \\ &= d'_U(v) \end{aligned}$$

For the converse, let us assume that for the quantum teams (Ω, τ) and (Ω', τ') their corresponding probability tables are the same: $d_U(v) = d'_U(v)$. Now of course also $[\phi_j]_{(\Omega,\tau),U_j} = [\phi_j]_{(\Omega',\tau'),U_j}$ for any formula ϕ_j with corresponding U_j. This readily implies that (Ω, τ) and (Ω', τ') are equivalent.

For the induction step let us assume that $(\Omega', \tau') \models \alpha$ iff $(\Omega, \tau) \models \alpha$ for an arbitrary formula α. This also means that $(\Omega', \tau') \not\models \alpha$ iff $(\Omega, \tau) \not\models \alpha$. So, by definition, this means that $(\Omega', \tau') \models \neg\alpha$ iff $(\Omega, \tau) \models \neg\alpha$.

Now also assume that for a formula β, we have $(\Omega', \tau') \models \beta$ iff $(\Omega, \tau) \models \beta$. Then we very easily see that $(\Omega', \tau') \models \alpha$ and $(\Omega', \tau') \models \beta$, iff $(\Omega, \tau) \models \alpha$ and $(\Omega, \tau) \models \beta$, which means, by definition that $(\Omega', \tau') \models \alpha \wedge \beta$ iff $(\Omega, \tau) \models \alpha \wedge \beta$.

So equivalent quantum teams have equal corresponding probability table, and quantum teams with the same probability tables are equivalent. □

Corollary 20. *In Bell scenarios of type $(n, k, 2)$, if (Ω', τ') is a reduced quantum team from (Ω, τ), then (Ω', τ') and (Ω, τ) are equivalent.*

Proof. Let (Ω', τ') be a reduced quantum team from (Ω, τ). By a lemma 14 (Ω', τ') and (Ω, τ) have the same corresponding probability table. By the previous lemma, they are equivalent. □

Definition 21. Two quantum teams (Ω_1, τ_1) and (Ω_2, τ_2) are *analogous* if the following hold:

- $|\Omega_1| = k\,|\Omega_2|$ or $|\Omega_2| = k\,|\Omega_1|$ for a natural number $k > 0$. Without loss of generality, let us assume in the following that (Ω_2, τ_2) is the bigger one.
- there is a function $f : \Omega_1 \to \{A \subseteq \Omega_2 : |A| = k\}, i \mapsto f(i) = \{j_1, \ldots, j_k\}$ such that $\tau_2(j_l) = \tau_1(i)$ for all $l \in \{1, \ldots k\}$ and $f(i) \cap f(i') = \emptyset$ when $i \neq i'$.

Informally, two quantum teams are analogous if the bigger one just consists of a number of copies of the smaller one, with the order of the rows possibly permuted.

Proposition 22. *In Bell scenarios of type $(n, k, 2)$ quantum teams are equivalent iff they are analogous.*

Proof. Let (Ω, τ) and (Ω', τ') be equivalent. Let $\{d_U\}_{U \in \mathcal{U}}$ and $\{d'_U\}_{U \in \mathcal{U}}$ be the associated probability tables of (Ω, τ) and (Ω', τ') respectively over the same cover $\mathcal{U}$. By lemma 19, the probability tables must be equal: $\{d_U\}_{U \in \mathcal{U}} = \{d'_U\}_{U \in \mathcal{U}}$. This means that for any measurement outcome v with corresponding measurement context $U \in \mathcal{U}$, we have:

$$\frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|} = \frac{|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{|\Omega'_U|}$$

We can make the denominators equal by finding a natural number k such that[4]:

$$\frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|} = \frac{k\,|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{k\,|\Omega'_U|}$$

So, $|\{i \in \Omega_U | \tau_U(i) = v\}| = k\,|\{i \in \Omega'_U | \tau'_U(i) = v\}|$. So for every element $i \in \Omega'_U$ such that for a given assignment v we have $\tau'_U(i) = v$, we can find k elements $i'_j \in \Omega_U$, $j \in \{0, \ldots, k-1\}$ such that $\tau_U(i) = v$. This constitutes the desired function $f : \Omega'_U \to \{A \subseteq \Omega_U : |A| = k\}$ with $i \mapsto \{j_1, \ldots, j_k\}$ where $\tau(j_l) = \tau'(i)$ for $l \in \{1, \ldots, k\}$ and $f(i) \neq f(i')$ for $i = i'$. So, (Ω, τ) and (Ω', τ') are analogous.

For the converse, assume that (Ω, τ) and (Ω', τ') are analogous. Then, without loss of generality, there is a k such that $|\Omega| = k\,|\Omega'|$ and there is, for any $U \in \mathcal{U}$, a function $f : \Omega'_U \to \{A \subseteq \Omega_U : |A| = k\}$ such that $i \mapsto \{j_1, \ldots, j_k\}$ where $\tau(j_l) = \tau'(i)$ for $l \in \{1, \ldots, k\}$ and $f(i) \neq f(i')$ for $i = i'$. Now we see that:

$$\begin{aligned} d'_U(v) &= \frac{|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{|\Omega'_U|} \\ &= \frac{k\,|\{i \in \Omega'_U | \tau'_U(i) = v\}|}{k\,|\Omega'_U|} \\ &= \frac{|\bigcup \{f(i) | \tau_U(i) = v\}|}{|\Omega_U|} \\ &= \frac{|\{i \in \Omega_U | \tau_U(i) = v\}|}{|\Omega_U|} = d_U(v) \end{aligned}$$

Having proved that the probability tables are the same, we can conclude, again using lemma 19, that (Ω, τ) and (Ω', τ') are equivalent, which concludes the proof. □

An open problem

QTL is capable of proving maximal violations of Bell inequalities in a type $(2, 2, 2)$ Bell scenario. This is undesirable, because these situations do not correspond to reality. An explicit open question in [HPV2016] is the problem of how to characterise precisely those QTL formulas that do correspond to the predictions of quantum mechanics. In an attempt to solve this problem, attention must be paid to an article by Boris S. Tsirelson [C1980]. In it he states the quantum analogue to Bell's theorem: a type $(2, 2, 2)$ Bell scenario is consistent with quantum mechanics iff it satisfies Tsirelson's inequality. A derivation can be found in the appendix. Tsirelson's inequality looks like:

[4] Without loss of generality, we assume that (Ω, τ) is the bigger quantum team.

$$A_1B_2 + A_2B_1 + A_1B_1 - A_2B_2 \leq 2\sqrt{2}$$

Here, A_i and B_j are quantum operators for the two agents respectively. A_iB_j is short for the tensor product $A_i \otimes B_j$ which is the correlation-operator of the observables A_i and B_j. It takes on the value of 1 when the measurement values correlate and it takes on the value of -1 when the measurement values anti-correlate. We can use this inequality to derive an inequality expressible in QTL. We first demand that A_1 signifies setting a for our first agent whom we call Alice. A_2 denotes Alice's setting a'. The other agent is called Bob and he has the settings b and b' corresponding to B_1 and B_2 respectively. Now A_1B_1 expresses correlation so it corresponds to the formula $a \leftrightarrow b$ and similarly for the other operators. From [AH2012] we quote the following correspondence between correlations E_i and probabilities p_i associated to propositions:

$$p_i = \frac{E_i + 1}{2}$$

Hence, we write Tsirelson's inequality as:

$$A_1B_2 + 1 + A_2B_1 + 1 + A_1B_1 + 1 - (A_2B_2 + 1) \leq 2\sqrt{2} + 2$$

And by dividing both sides by two, we get:

$$\frac{A_1B_2 + 1}{2} + \frac{A_2B_1 + 1}{2} + \frac{A_1B_1 + 1}{2} - \frac{A_2B_2 + 1}{2} \leq \sqrt{2} + 1$$

Which now actually means:

$$[a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} - [a' \leftrightarrow b']_{U_3} \leq \sqrt{2} + 1$$

Where, perhaps superfluously, we used the following definitions: $U_0 = \{a, b'\}$, $U_1 = \{a', b\}$, $U_2 = \{a, b\}$, and $U_3 = \{a', b'\}$. By adding 1 to both sides and by noting that $1 - [\phi]_X = [\neg\phi]_X$, we arrive at:

$$[a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \leq \sqrt{2} + 2$$

Which is the inequality that should be satisfied. Note that the set of propositional logic formulas appearing in the expression is collectively contradictory, which means that the corresponding Bell inequality is:

$$[a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \leq 3$$

So the interval $[3, 2+\sqrt{2}]$ is 'where the magic happens'. This expression cannot readily be formulated in QTL, because QTL only accounts for rational inequalities, while $\sqrt{2} \notin \mathbb{Q}$, as has been long known since the ancient Greeks.

Lemma 23. *For a type* $(2, 2, 2)$ *Bell scenario, there exists a complete axiomatisation of the QTL formulas that are valid in quantum teams that correspond to quantum reality. This set is recursive.*

Proof. Let $(a_i)_{i\in\mathbb{N}}$ be the infinite sequence of decimals of $\sqrt{2}$. We define the number s_i recursively:

1. $s_0 = 1$
2. $s_{i+1} = s_i + a_i \times 10^{-i}$

It denotes the partial decimal expansion of $\sqrt{2}$, its values are: $(1; 1.4; 1.41; 1.414, ...)$. Using this sequence, we define the sequence t_i:

1. $t_0 = 4$
2. $t_{i+1} = 2 + s_{i+1} + 10^{-i}$

So, these values are: $(4; 3.5; 3.42; 3.415, ...)$

The sequence (t_i) approaches $2 + \sqrt{2}$ from above, starting from 4.

We now define:

$$\phi = [a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \leq 0$$

$$\psi_i = \neg\left([a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \geq t_i\right)$$

Note now that ϕ and ψ_i are not QTL formulas in the strict sense of the word. Expressing the inequalities as above, however, makes them easier to work with. We define $\Gamma = \{\psi_i | i \in \mathbb{N}\} \cup \{\phi\}$. Now Γ axiomatises completely the set of QTL formulas consistent with quantum mechanics. To see this, let us look at a QTL formula χ that does not violate Tsirelson's inequality. It has the form:

$$\chi = [a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \leq c$$

for some $c \in \mathbb{Q}$. Note now that $c \geq 0$, so ϕ implies χ and we see that the QTL formulas consistent with quantum mechanics are proved in quite a trivial way.

If, on the other hand, χ is inconsistent with quantum mechanics, it must violate Tsirelson's inequality. That is, it must have the form:

$$\chi = [a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \geq c$$

for a certain $c \in \mathbb{Q}$ with $c > 2 + \sqrt{2}$.

Because the sequence $(t_i)_{i\in\mathbb{N}}$ is strictly decreasing, there must be an $n \in \mathbb{N}$ for which $t_n < c$. For this n, we have that χ implies:

$$[a \leftrightarrow b']_{U_0} + [a' \leftrightarrow b]_{U_1} + [a \leftrightarrow b]_{U_2} + [\neg(a' \leftrightarrow b')]_{U_3} \geq t_n$$

which contradicts $\psi_n \in \Gamma$. So QTL formulas inconsistent with quantum mechanics contradict our axiomatisation. So the set Γ completely axiomatises the desired set of QTL formulas and this set is recursive. □

4 Demonstrating Contextuality

In this section, we prove that non-contextual quantum teams reduce to multi-teams. The reduction procedure is therefore a tool to detect quantum behaviour in a given Bell scenario.

Definition 24. In a Bell scenario of type (n, k, l), let $\{d_U\}_{U \in \mathcal{U}}$ be a probability table. Let U and V be measurement contexts with $U \cap V = p$. $\{d_U\}_{U \in \mathcal{U}}$ satisfies *non-signalling* if $d_U(p \mapsto v) = d_V(p \mapsto v)$ for $v \in \{0, ..., l-1\}$.

Informally, it means that the probabilities that one agent measures are independent of the measurement settings of the other agent(s). If this dependence actually existed, it could be used to send signals from one agent to another without any assumptions (and hence restrictions) on the signals' speed. For example, see table 9. Alice, by keeping her measurement setting fixed, can receive signals from Bob upon his changing his measurement setting. Bob can likewise receive signals from Alice.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	1	0	0	0
(a,b')	0	0	1	0
(a',b)	0	1	0	0
(a',b')	0	0	0	1

Table 9: *A Bell scenario of type* $(2, 2, 2)$ *in which the agents are called Alice and Bob. This probability table violates non-signalling. It can be used by Alice and Bob to send signals at any speed.*

Lemma 25. *Every non-contextual probability table in a Bell scenario of type* $(n, k, 2)$ *satisfies non-signalling.*

Proof. Let a probability table $\{d_U\}_{U \in \mathcal{U}}$ be given, such that it is non-contextual, i.e. any Bell inequality is satisfied instead of violated. We can select two measurement contexts $U, V \in \mathcal{U}$ with a propositional symbol p in common. We define formulas $\phi_U = p$ and $\phi_V = \neg p$ pertaining to U and V respectively. The formulas contradict each other, so they obey, rather than violate, the logical Bell inequality:

$$[\phi_U]_U + [\phi_V]_V = [\phi_U \wedge \phi_V]_{U \cup V} \leq 2 - 1 = 1$$

Similarly, we can define $\phi'_U = \neg p$ and $\phi'_V = p$ and get $[\phi'_U]_X + [\phi'_V]_X \leq 1$. We also have the following:

$$\begin{aligned} [\phi_U]_U + [\phi'_U]_U + [\phi_V]_V + [\phi'_V]_V &= [\phi_U \wedge \phi'_U]_U + [\phi_V \wedge \phi'_V]_V \\ &= [\top]_U + [\top]_V \\ &= 2 \end{aligned}$$

So we can conclude that:

$$\begin{aligned} [\phi_U]_U + [\phi_V]_V &= 1 \\ [\phi'_U]_U + [\phi'_V]_V &= 1 \end{aligned}$$

And combining this again with the facts that:

$$\begin{aligned} [\phi_U]_U + [\phi'_U]_U &= 1 \\ [\phi_V]_V + [\phi'_V]_V &= 1 \end{aligned}$$

we get:

$$\begin{aligned} [\phi_U]_U &= [\phi'_V]_V \\ [\phi_V]_V &= [\phi'{}_U]_U \end{aligned}$$

No assumptions were made on the properties of p, so we can conclude that the probability of p taking on a specific value v is equal for both measurement contexts in which p appears and is therefore independent of the measurement setting(s) of the other agent(s). Also, no assumptions were made on the number of measurement settings. The argument can be repeated for every pair of agents. □

Proposition 26. *For every non-contextual probability table $\{d_U\}_{U\in\mathcal{U}}$ with rational probabilities in a Bell scenario of type $(n, k, 2)$, there is a quantum team (Ω, τ) of size $D \times N$ for some D such that there are D rows for every measurement context and $\{d_U\}_{U\in\mathcal{U}}$ is its associated probability table.*

Proof. Let a probability table $\{d_U\}_{U\in\mathcal{U}}$ be given, such that it is non-contextual. We write all fractions $d_{U_i}(v)$ such that they have the same denominator D. We are going to construct a quantum team of size D that has $\{d_U\}_{U\in\mathcal{U}}$ as the associated probability table. We construct the rows of this quantum team one by one. We pick a measurement outcome $f_0 = \{\vec{p} \to \vec{v_p}\}$, pertaining to the measurement context U_0, that has non-zero probability $d_{U_0}(f_0) = \frac{a}{D}$. Now we define $\tau_{U_0}(0) = f_0$ and define:

$$d_U^0(f) = \begin{cases} \frac{a-1}{D} & \text{if } U = U_0 \text{ and } f = f_0 \\ d_U(f) & \text{otherwise} \end{cases}$$

d_U^0 Still contains fractions, but is not a probability distribution anymore; from now on it can be considered as an accounting tool used to construct the desired quantum team. Informally speaking, compared to $\{d_U\}_{U\in\mathcal{U}}$, the numerator of the probability corresponding to the chosen measurement outcome has been lowered by 1. We can repeat this a number of times, but process stops after $D \times N$ steps, when all values in $\left\{d_U^{D\times N-1}\right\}_{U\in\mathcal{U}}$ are equal to zero. The resulting quantum team (Ω, τ), where $\Omega = \{0, \ldots, D \times N - 1\}$ and $\tau = \left\{\tau_{U_0}(0), \ldots, \tau_{U_{D\times N-1}}(D \times N - 1)\right\}$ is the desired quantum team. □

Lemma 27. *In a Bell scenario of type $(n, k, 2)$, every non-contextual quantum team (Ω, τ) of size $D \times N$ such that there are D rows for each measurement context reduces to a multi-team.*

Proof. Let the finite set $f_i = \{a_{j_i} \mapsto v_a^i, b_{j_i} \mapsto v_b^i, \ldots\}$ be a measurement outcome where a_i is some measurement setting with result v_a^i for agent A and similarly for $B, C, \ldots$. The corresponding measurement context $U_i = \text{dom}(f_i)$. f_i And f_j can be *linked* if $U_i \cap U_j \neq \emptyset$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so they must have some measurement setting that they agree upon and they agree on all of those. In a Bell scenario with non-signalling, for any measurement outcome f_i and any measurement setting $p \in \text{dom}(f_i)$, there is a measurement setting f_j, with $i \neq j$ such that $f_i(p) = f_j(p)$. Note that it is possible that these measurement settings don't link. Picking a measurement context f_i and a measurement setting $p \in U_i$, we can define the set $S_1 = \{f_j | f_j \text{ links to } f_i\}$ with corresponding domain $D_1 = \bigcup_{f_j \in S_1} U_j$. Furthermore, we can define $S_{n+1} = \{f_j | f_j \text{ links to a measurement setting } f_i \in S_n\}$ with domain D_{n+1}. If our quantum team (Ω, τ) is of size $D \times N$ (note the different D here), then for sure $S_{D\times N+1} = S_{D\times N}$, because there are no more measurement outcomes. We call $S_{D\times N}$ a *quilt.*

There are two cases: $D_{D\times N} \neq X$ or $D_{D\times N} = X$.

In the former case, there exist finitely many other quilts Q_i such that $D_{D\times N} \cup \bigcup_i Q_i = X$. To construct a formula violating a logical Bell inequality, we restrict each quilt Q to Q^R such that they have non-overlapping domains. Each Q^R is an assignment on its domain and has therefore an associated propositional logic formula defining this assignment. We define this formula thus: $\phi_{Q^R} = \bigwedge_{p \in D_{Q^R},\, Q^R(f)=1} p \wedge \bigwedge_{p \in D_{Q^R},\, Q^R(f)=0} \neg p$. Now the formula $\bigwedge_{Q^R} \phi_{Q^R}$ is a contradiction and violates the logical Bell inequality $\sum_{Q^R} \left[\phi_{Q^R}\right]_{(\Omega,\tau)} \leq M - 1$ maximally with M the number of (restricted) quilts, as $\left[\phi_{Q^R}\right]_{(\Omega,\tau)} = 1$ for every restricted quilt Q^R. This case therefore, is impossible for a non-contextual quantum team.

In the case that $D_{D\times N} = X$, we can have yet again two cases. The first is where $S_{D\times N}$ is a global section; in which case we are done. The second case is one where $S_{D\times N}$ is not a global section. This means that there is a propositional variable p appearing in at least two measurement contexts U_i and U_j, such that $f_i(p) \neq f_j(p)$. Due to non-signalling, there must be measurement outcomes f_j' and f_i' pertaining to measurement contexts U_j and U_i respectively such that $f_i(p) = f_j'(p)$ and $f_j(p) = f_i'(p)$. We can define $S_1^1 = \{f | f \text{ links to } f_j'\}$, $S_1^2 = \{f | f \text{ links to } f_i'\}$ and $S_{n+1}^1 = \{f | f \text{ links to a measurement setting } f_i \in S_n^1\}$ and S_{n+1}^2 in similar fashion. Now there is a measurement setting $q \in \left(D_{n+1}^i - D_n^i\right)$ such that $S_{n+1}^i(q) \neq S_{D\times N}$ or else we would have a global section.[5] So, continuing our quilting in this way, which we can do because of non-signalling, we must arrive at the conclusion that for some n and m, there is an $r \in S_n^1 \cap S_m^2$. So there is a sequence (f_k) such that:

- $f_0 = f_i$
- $f_1 = f_j'$
- $f_{l+1} \in S_{l+1}^1$, which links to f_l with $f_l \in S_l^1$ for $l \in \{1, \ldots, n-1\}$
- $f_n = r$

[5]Two actually: one in which $p \mapsto 0$ and one in which $p \mapsto 1$.

- $f_{l+1} \in S^2_{m-l}$, which links to f_l with $f_l \in S^2_{m-l}$ for $l \in \{n+1, \dots, n+m\}$
- $f_{n+m+1} = f_j$
- $f_{l+1} \in S_{D\times N}$, which links to f_l with $f_l \in S_{D\times N}$ for $l \in \{n+m+1, \dots, 2(n+m)-1\}$
- $f_{2(m+n)} = f_i$

This sequence gives rise to the violation of another logical Bell inequality. Let ψ_k be a propositional logical formula defining f_k, that is $\psi_k = \bigwedge_{p\in U_k,\ f(p)=1} p \wedge \bigwedge_{p\in U_k,\ f(p)=0} \neg p$. For every valuation f_l there is a counterpart $\overline{f_l}$ such that there is a $q \in \mathrm{dom}(f_l) \cap \mathrm{dom}(\overline{f_l})$ with $f_l(q) \neq \overline{f_l}(q)$. Note that $\neg\psi_k$ corresponds to $\overline{f_k}$. We define $\chi_l = \psi_l \vee \neg\psi_l$. The logical Bell inequality $\sum_{l<n+m} [\chi l]_{(\Omega,\tau)} = m+n \geq m+n-1$, where the numbering is such as to avoid counting twice, is again violated maximally. This case is, therefore, also impossible in a non-contextual situation. Hence, we can always find a global section.

Any reduction procedure reduces the non-contextual quantum team to a multi-team, because we can always find a global section. Note that the requirements pertaining to the size of (Ω, τ) and the size of its measurement contexts U was necessary to avoid ending up with undetermined rows for which no global section can be found on the basis of there being to few of them to constitute a measurement cover.

□

Proposition 28. *Let (Ω, τ) be a quantum team in a type $(n, k, 2)$ Bell scenario such that there are D rows for every measurement context. (Ω, τ) is non-contextual iff its reduced form is a multi-team.*

Proof. The direction from left to right is given by lemma 27. For the other direction, we use the fact expressed in Corollary 20 that QTL properties do not change during a reduction: all Bell inequalities that are violated, remain so. Assuming the reduced form is a multi-team, which satisfies all Bell inequalities, (Ω, τ) must satisfy all Bell inequalities as well and must hence be non-contextual. □

5 Conclusion

We defined a reduction procedure for quantum teams as an attempt to explain Bell scenarios of a certain type (n, k, l) in classical terms, i.e. with hidden variables. This procedure is not unique, but always yields a hybrid team. If this hybrid team is a multi-team, the original quantum team displayed no contextuality. Contextuality is, however, present when the resulting hybrid team is non-trivial. The procedure preserves the logical properties of the team in terms of QTL. Moreover, equivalence between two quantum teams in this logic coincides with a structural resemblance called analogousness. An attempt has been made to solve an open problem, namely the axiomatisation of QTL formulas that do not violate quantum mechanics. This set is proved to be recursive, but only applies to Bell scenarios of type $(2, 2, 2)$.

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Appendix 1: Constructing the Probability Table

For the sake of completeness, this appendix contains the derivation of the Bell probability table. It can be calculated from quantum mechanics in the following way. We assume a two-level quantum system like the spin of an electron. We assume both Alice and Bob to measure their electrons under two different angles, to be specified later. The probabilities are calculated by taking inner products of the operators with the quantum state. We work in the z-basis. In that basis, corresponding to a measurement of the electron along the x-axis, we have the projections: $\langle S_X;\uparrow| = \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right)$ for up and $\langle S_X;\downarrow| = \frac{1}{\sqrt{2}}\left(\langle\uparrow| - \langle\downarrow|\right)$ for down. The projection corresponding to a measurement making an angle of $\frac{\pi}{3}$ with the x-axis, looks like: $\langle S_\measuredangle;\uparrow| = \frac{1}{\sqrt{2}}\left(\langle\uparrow| + e^{-i\frac{\pi}{3}}\langle\downarrow|\right) = \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow|\right)$ for up and $\langle S_\measuredangle;\downarrow| = \frac{1}{\sqrt{2}}\left(\langle\uparrow| - e^{-i\frac{\pi}{3}}\langle\downarrow|\right) = \frac{1}{\sqrt{2}}\left(\langle\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow|\right)$ for a down.

When two agents (for example Alice and Bob) are measuring similar two-level quantum systems, the 'combined' projection of the system as a whole is constructed by the tensor product:

$$\begin{aligned}
\langle S_X, S_X; 1, 1| &= \langle S_X;\uparrow| \otimes \langle S_X;\uparrow| \\
&= \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right) \otimes \frac{1}{\sqrt{2}}\left(\langle\uparrow| + \langle\downarrow|\right) \\
&= \frac{1}{2}\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow| + \langle\downarrow\downarrow|\right)
\end{aligned}$$

Note that 0 corresponds to $\downarrow$ and 1 corresponds to $\uparrow$.

This yields 16 different operators, one for each measurement (2 for Alice and 2 for Bob, so 4 in total) and measurement outcome (up and down, both for Alice and Bob).

$$\begin{aligned}
\langle S_X, S_X; 1, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow| + \langle\downarrow\downarrow|\right) \\
\langle S_X, S_X; 1, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right) \\
\langle S_X, S_X; 0, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| - \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right) \\
\langle S_X, S_X; 0, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| - \langle\downarrow\uparrow| + \langle\downarrow\downarrow|\right) \\
\langle S_X, S_\measuredangle; 1, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| + \langle\downarrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_X, S_\measuredangle; 1, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_X, S_\measuredangle; 0, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| - \langle\downarrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_X, S_\measuredangle; 0, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| - \langle\downarrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_X; 1, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_X; 1, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_X; 0, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_X; 0, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_\measuredangle; 1, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_\measuredangle; 1, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| - \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_\measuredangle; 0, 1| &= \frac{1}{2}\left(\langle\uparrow\uparrow| + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| - \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right) \\
\langle S_\measuredangle, S_\measuredangle; 0, 0| &= \frac{1}{2}\left(\langle\uparrow\uparrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\uparrow\downarrow| - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\uparrow| + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)\langle\downarrow\downarrow|\right)
\end{aligned}$$

When measuring the entangled state:

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$$

We get probabilities, according to the formula:

$$P = |\langle S_1, S_2; m_1, m_2|\ \alpha\rangle|^2$$

Firstly, the bra(c)ket expressions become:

$$\begin{aligned}
\langle S_X, S_X; 1, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}} \\
\langle S_X, S_X; 1, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}(1-1) = 0 \\
\langle S_X, S_X; 0, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}(1-1) = 0 \\
\langle S_X, S_X; 0, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}} \\
\langle S_X, S_\measuredangle; 1, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3-\sqrt{3}i}{4\sqrt{2}} \\
\langle S_X, S_\measuredangle; 1, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_X, S_\measuredangle; 0, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_X, S_\measuredangle; 0, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3-\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_X; 1, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3-\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_X; 1, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_X; 0, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_X; 0, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3-\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_\measuredangle; 1, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1-\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_\measuredangle; 1, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_\measuredangle; 0, 1|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1-\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{3+\sqrt{3}i}{4\sqrt{2}} \\
\langle S_\measuredangle, S_\measuredangle; 0, 0|\ \alpha\rangle &= \frac{1}{2\sqrt{2}}\left(1+\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)\right) = \frac{1-\sqrt{3}i}{4\sqrt{2}}
\end{aligned}$$

And secondly, the probabilities become:

$$
\begin{aligned}
P(a,b;0,0) &= \frac{1}{2} \\
P(a,b;1,0) &= 0 \\
P(a,b;0,1) &= 0 \\
P(a,b;1,1) &= \frac{1}{2} \\
P(a,b';0,0) &= \left|\frac{3-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{3-\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{3+\sqrt{3}i}{4\sqrt{2}}\right) = \frac{9+3}{32} = \frac{3}{8} \\
P(a,b';1,0) &= \left|\frac{1+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{1+\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{1-\sqrt{3}i}{4\sqrt{2}}\right) = \frac{1+3}{32} = \frac{1}{8} \\
P(a,b';0,1) &= \left|\frac{1+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{1+\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{1-\sqrt{3}i}{4\sqrt{2}}\right) = \frac{1+3}{32} = \frac{1}{8} \\
P(a,b';1,1) &= \left|\frac{3-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{3-\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{3+\sqrt{3}i}{4\sqrt{2}}\right) = \frac{9+3}{32} = \frac{3}{8} \\
P(a',b;0,0) &= \left|\frac{3-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{3-\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{3+\sqrt{3}i}{4\sqrt{2}}\right) = \frac{9+3}{32} = \frac{3}{8} \\
P(a',b;1,0) &= \left|\frac{1+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{1+\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{1-\sqrt{3}i}{4\sqrt{2}}\right) = \frac{1+3}{32} = \frac{1}{8} \\
P(a',b;0,1) &= \left|\frac{1+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{1+\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{1-\sqrt{3}i}{4\sqrt{2}}\right) = \frac{1+3}{32} = \frac{1}{8} \\
P(a',b;1,1) &= \left|\frac{3-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \left(\frac{3-\sqrt{3}i}{4\sqrt{2}}\right)\left(\frac{3+\sqrt{3}i}{4\sqrt{2}}\right) = \frac{9+3}{32} = \frac{3}{8} \\
P(a',b';0,0) &= \left|\frac{1-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \frac{1+3}{32} = \frac{1}{8} \\
P(a',b';1,0) &= \left|\frac{3+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \frac{9+3}{32} = \frac{3}{8} \\
P(a',b';0,1) &= \left|\frac{3+\sqrt{3}i}{4\sqrt{2}}\right|^2 = \frac{9+3}{32} = \frac{3}{8} \\
P(a',b';1,1) &= \left|\frac{1-\sqrt{3}i}{4\sqrt{2}}\right|^2 = \frac{1+3}{32} = \frac{1}{8}
\end{aligned}
$$

Thus, we get precisely the Bell probability table as in table 1.2.

Appendix 2: Calculating Tsirelson's bound

For the sake of completeness, a derivation of Tsirelson's inequality is given. It is an inequality that is precisely satisfied (as opposed to being violated) by systems in a type $(2,2,2)$ Bell scenario that agree with quantum mechanics. First we calculate some conspicuously looking expressions:

$$\begin{aligned}
\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1-B_1)+A_2-B_2\right)^2 &= \frac{\sqrt{2}-1}{8}(\left(3+2\sqrt{2}\right)\left(A_1^2-2A_1B_1+B_1^2\right)+A_2^2+B_2^2-2A_2B_2\\
&\quad+\left(\sqrt{2}+1\right)(A_1A_2-B_1A_2-A_1B_2+B_1B_2)\\
&\quad+\left(\sqrt{2}+1\right)(A_2A_1-A_2B_1-B_2A_1+B_2B_1))\\
&= \frac{1+\sqrt{2}}{8}\left(A_1^2-2A_1B_1+B_1^2\right)\\
&\quad-\frac{\sqrt{2}-1}{4}A_2B_2\\
&\quad+\frac{\sqrt{2}-1}{8}\left(A_2^2+B_2^2\right)\\
&\quad+\frac{1}{8}\left(A_1A_2+A_2A_1-2\left(A_1B_2+A_2B_1\right)+B_1B_2+B_2B_1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1-B_2)-A_2-B_1\right)^2 &= \frac{\sqrt{2}-1}{8}(\left(3+2\sqrt{2}\right)\left(A_1^2-2A_1B_2+B_2^2\right)+A_2^2+B_1^2+2A_2B_1\\
&\quad+\left(\sqrt{2}+1\right)(-A_1A_2+B_2A_2-A_1B_1+B_2B_1)\\
&\quad+\left(\sqrt{2}+1\right)(-A_2A_1+A_2B_2-B_1A_1+B_1B_2))\\
&= \frac{1+\sqrt{2}}{8}\left(A_1^2-2A_1B_2+B_2^2\right)\\
&\quad+\frac{\sqrt{2}-1}{4}A_2B_1\\
&\quad+\frac{\sqrt{2}-1}{8}\left(A_2^2+B_1^2\right)\\
&\quad+\frac{1}{8}\left(-A_1A_2-A_2A_1-2\left(A_1B_1-A_2B_2\right)+B_2B_1+B_1B_2\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2-B_1)+A_1+B_2\right)^2 \quad &= \quad \frac{\sqrt{2}-1}{8}(\left(3+2\sqrt{2}\right)\left(A_2^2-2A_2B_1+B_1^2\right)+A_1^2+B_2^2+2A_1B_1 \\
&+\left(\sqrt{2}+1\right)(A_2A_1-B_1A_1+A_2B_2-B_1B_2) \\
&+\left(\sqrt{2}+1\right)(A_1A_2-A_1B_1+B_2A_2-B_2B_1)) \\
&= \quad \frac{1+\sqrt{2}}{8}\left(A_2^2-2A_2B_1+B_1^2\right) \\
&+\frac{\sqrt{2}-1}{4}A_1B_1 \\
&+\frac{\sqrt{2}-1}{8}\left(A_1^2+B_2^2\right) \\
&+\frac{1}{8}(A_2A_1+A_1A_2-2(-A_2B_2+A_1B_1)-B_1B_2-B_2B_1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2+B_2)-A_1-B_1\right)^2 \quad &= \quad \frac{\sqrt{2}-1}{8}(\left(3+2\sqrt{2}\right)\left(A_2^2+2A_2B_2+B_2^2\right)+A_2^2+B_1^2+2A_1B_2 \\
&+\left(\sqrt{2}+1\right)(-A_2A_1-B_2A_1-A_2B_1-B_2B_1) \\
&+\left(\sqrt{2}+1\right)(-A_1A_2-A_1B_2-B_1A_2-B_1B_2)) \\
&= \quad \frac{1+\sqrt{2}}{8}\left(A_2^2+2A_2B_2+B_2^2\right) \\
&+\frac{\sqrt{2}-1}{4}A_1B_2 \\
&+\frac{\sqrt{2}-1}{8}\left(A_1^2+B_1^2\right) \\
&+\frac{1}{8}(-A_2A_1-A_1A_2-2(+A_2B_1+A_1B_2)-B_2B_1-B_1B_2)
\end{aligned}
$$

Summing these expressions gives an interesting, and very concise, result:

$$
\begin{aligned}
&\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1-B_1)+A_2-B_2\right)^2 \\
&+\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1-B_2)-A_2-B_1\right)^2 \\
&+\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2-B_1)+A_1+B_2\right)^2 \\
&+\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2+B_2)-A_1-B_1\right)^2 \quad = \quad \frac{1+\sqrt{2}}{8}\left(A_1^2-2A_1B_1+B_1^2\right) \\
&\qquad -\frac{\sqrt{2}-1}{4}A_2B_2 \\
&\qquad +\frac{\sqrt{2}-1}{8}\left(A_2^2+B_2^2\right) \\
&\qquad +\frac{1}{8}\left(A_1A_2+A_2A_1-2\left(A_1B_2+A_2B_1\right)+B_1B_2+B_2B_1\right) \\
&\qquad +\frac{1+\sqrt{2}}{8}\left(A_1^2-2A_1B_2+B_2^2\right) \\
&\qquad +\frac{\sqrt{2}-1}{4}A_2B_1 \\
&\qquad +\frac{\sqrt{2}-1}{8}\left(A_2^2+B_1^2\right) \\
&\qquad +\frac{1}{8}\left(-A_1A_2-A_2A_1-2\left(A_1B_1-A_2B_2\right)+B_2B_1+B_1B_2\right) \\
&\qquad +\frac{1+\sqrt{2}}{8}\left(A_2^2-2A_2B_1+B_1^2\right) \\
&\qquad +\frac{\sqrt{2}-1}{4}A_1B_1 \\
&\qquad +\frac{\sqrt{2}-1}{8}\left(A_1^2+B_2^2\right) \\
&\qquad +\frac{1}{8}\left(A_2A_1+A_1A_2-2\left(-A_2B_2+A_1B_1\right)-B_1B_2-B_2B_1\right) \\
&\qquad +\frac{1+\sqrt{2}}{8}\left(A_2^2+2A_2B_2+B_2^2\right) \\
&\qquad +\frac{\sqrt{2}-1}{4}A_1B_2 \\
&\qquad +\frac{\sqrt{2}-1}{8}\left(A_1^2+B_1^2\right) \\
&\qquad +\frac{1}{8}\left(-A_2A_1-A_1A_2-2\left(A_2B_1+A_1B_2\right)-B_2B_1-B_1B_2\right) \\
&= \left(A_1^2+A_2^2+B_1^2+B_2^2\right)\left(\frac{1+\sqrt{2}}{8}+\frac{1+\sqrt{2}}{8}+\frac{\sqrt{2}-1}{8}+\frac{\sqrt{2}-1}{8}\right) \\
&\qquad +\frac{\sqrt{2}-1}{4}\left(-A_2B_2+A_2B_1+A_1B_1+A_1B_2\right) \\
&\qquad -\frac{2+2\sqrt{2}}{8}\left(A_1B_1+A_1B_2+A_2B_1-A_2B_2\right) \\
&\qquad -\frac{4}{8}\left(A_1B_2+A_2B_1+A_1B_1-A_2B_2\right) \\
&= \left(A_1^2+A_2^2+B_1^2+B_2^2\right)\left(\frac{1}{\sqrt{2}}\right) \\
&\qquad -\left(A_1B_2+A_2B_1+A_1B_1-A_2B_2\right)
\end{aligned}
$$

In summary, we see that:

$$\begin{aligned} A_1B_2 + A_2B_1 + A_1B_1 - A_2B_2 &= \frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) \\ &-\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1 - B_1) + A_2 - B_2\right)^2 \\ &-\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_1 - B_2) - A_2 - B_1\right)^2 \\ &-\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2 - B_1) + A_1 + B_2\right)^2 \\ &-\frac{\sqrt{2}-1}{8}\left(\left(\sqrt{2}+1\right)(A_2 + B_2) - A_1 - B_1\right)^2 \end{aligned}$$

The second to fifth term are each negative, so we can derive the following inequality:

$$\begin{aligned} A_1B_2 + A_2B_1 + A_1B_1 - A_2B_2 &\leq \frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) \\ &\leq \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

Where we used the fact that the expected value of a squared operator equals unity.