CHALMERS | GÖTEBORGS UNIVERSITET

Always Look on the Positive-Definite Side of Life

Positive-Definite Distributions and the Abel Transform

A thesis in fulfilment of a Masters degree in Mathematics at the University of Gothenburg

Mattias Byléhn

Department of Mathematical Sciences
CHALMERS TEKNISKA HÖGSKOLA
GÖTEBORGS UNIVERSITET
Göteborg, Sweden 2020

Always Look on the Positive-Definite Side of Life

Positive-Definite Distributions and the Abel Transform

A thesis in fulfilment of a Masters degree in Mathematics at the University of Gothenburg

Mattias Byléhn

Supervisor: Michael Björklund
Examiner: Genkai Zhang

Department of Mathematical Sciences
CHALMERS TEKNISKA HÖGSKOLA
GÖTEBORGS UNIVERSITET
Göteborg, Sweden 2020

Abstract

This thesis concerns distributions on $\mathbb{R}^n$ with the property of being positive-definite relative to a finite subgroup of the orthogonal group $\mathrm{O}(n)$. We construct examples of such distributions as the inverse Abel transform of Dirac combs on the geometries of Euclidean space $\mathbb{R}^n$ and the real- and complex hyperbolic plane $\mathbb{H}^2$, $\mathbb{H}^2_{\mathbb{C}}$. In the case of $\mathbb{R}^3$ we obtain Guinand's distribution as the inverse Abel transform of the Dirac comb on the standard lattice $\mathbb{Z}^3 < \mathbb{R}^3$. The main theorem of the paper is due to Bopp, Gelfand-Vilenkin and Krein, stating that a distribution on $\mathbb{R}^n$ is positive-definite relative to a finite subgroup $W < \mathrm{O}(n)$ if and only if it is the Fourier transform of a positive W-invariant Radon measure on

$$\left\{z \in \mathbb{C}^n : \overline{z} \in W.z\right\} \subset \mathbb{C}^n.$$

We present Bopp's proof of this theorem using a version of the Plancherel-Godement theorem for complex commutative $*$-algebras.

Keywords: *Poisson summation, positive-definite distributions, Abel transform, Guinand's distribution, relatively positive-definite distributions, Krein's theorem, Krein measures.*

Contents

1 Introduction

In this thesis we study positive-definite distributions and construct some examples of these using the inverse Abel transform in the context of generalized Poisson summation. Poisson summation formulae have proven to be an important tool in modern number theory and harmonic analysis, with one main example being the Selberg trace formula, and they appear for instance as a construction in the mathematical theory of *diffraction*. The trace formula in particular connects the unitary representation theory of a group and the geometry of lattices in it. One can parametrize these representations up to unitary equivalence by so called *positive-definite functions*. We study a weaker form of positive-definiteness that we refer to as *relative positive-definiteness* and provide a classification of distributions with this property, following the groundbreaking work of Krein, Gelfand-Vilenkin-Shilov and Bopp.

1.1 Motivation

To highlight the main ideas of this paper and their significance, we give a short overview of some of the main aspects of the mathematical diffraction developed in [3, 4, 5]. In the next subsections we give a description of how one obtains diffraction measures using the Abel transform of positive-definite measures.

1.1.1 Autocorrelation Measures

The motivation for this thesis stems from the theory of diffraction on locally compact homogeneous metric spaces, developed in [4]. In the general setting, when $X = G/K$ is such a homogeneous space, one considers for a translation bounded measure μ_o its *hull*

$$\Omega_o = \overline{G.\mu_o} \subset \mathrm{Radon}^+(X).$$

It is a compact space with a jointly continuous action of G on it, so one can look for G-invariant, and more specifically, ergodic measures $\nu \in \mathrm{Prob}_G(\Omega_o)$ with respect to the action. In many interesting cases the system (Ω_o, G, ν) is actually uniquely ergodic. If ν is an ergodic measure for the system (Ω_o, G) then we can associate to it an *autocorrelation measure* η_ν on G by

$$\eta_\nu(f * f^*) = \int_{\Omega_o} |\mu(f)|^2 \, d\nu(\mu).$$

This measure is by definition positive-definite, see section 3. A simple/trivial example that connects to Poisson summation is to take μ_o to be the Dirac comb $\delta_{\mathbb{Z}^n}$ on the standard lattice $\mathbb{Z}^n < \mathbb{R}^n$. The hull can be identified with the flat n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and the unique ergodic measure ν on it is the Lebesgue measure. Moreover, the autocorrelation measure of ν is by Poisson summation identified with $\delta_{\mathbb{Z}^n}$. If we consider $\mathbb{R}^n$ as the homogeneous space $(\mathrm{O}(n) \ltimes \mathbb{R}^n)/\mathrm{O}(n)$, then we can define the *Abel transform* $\mathscr{A}$ on radial/left-$\mathrm{O}(n)$-invariant test functions by

$$\mathscr{A} f(t) = \int_{\mathbb{R}^{n-1}} f(t, y) \, dy.$$

It defines a $*$-isomorphism from radial test functions on $\mathbb{R}^n$ onto even test functions on $\mathbb{R}$, and it extends to Schwartz functions. Dualizing this map to distributions, we define the *autocorrelation distribution* of ν by

$$\xi_\nu = \mathscr{A}^{-1} \eta_\nu.$$

It is positive-definite with respect to *even* functions, and in the case of $\mathbb{R}^3 = (\mathrm{O}(3) \ltimes \mathbb{R}^3)/\mathrm{O}(3)$ with the measure $\delta_{\mathbb{Z}^3}$ we observe in section 4 that it is (the derivative of) *Guinand's distribution*

$$\sigma_3(\varphi) = -2\varphi'(0) + \sum_{m=1}^{\infty} \frac{r_3(m)}{\sqrt{m}} (\varphi(\sqrt{m}) - \varphi(-\sqrt{m})).$$

This distribution has most notably been studied by Guinand in [11] and Meyer in [14], and it satisfies

$$\widehat{\sigma}_3 = -i\sigma_3 .$$

We derive this non-trivial Poisson summation formula using the inverse Abel transform. One interpretation of this formula is that we push the ordinary Poisson summation formula $\widehat{\delta}_{\mathbb{Z}^n} = \delta_{\mathbb{Z}^n}$ on $\mathbb{R}^n$ down to $\mathbb{R}$ when we apply the inverse Abel transform.

1.1.2 Relatively Positive-Definite Distributions

While the autocorrelation distribution ξ_ν in the Euclidean case turns out to be positive-definite, it is at first glance only positive-definite with respect to *even* test functions. Generally, if $W < \mathrm{GL}_n(\mathbb{R})$ is a subgroup then a distribution ξ on $\mathbb{R}^n$ is *W-positive-definite* if it is positive-definite with respect to all W-invariant test functions. The central result in this thesis is *Krein's theorem*, which realizes relatively positive-definite distributions in terms of measures.

Theorem 1.1. *(Bopp-Gelfand-Vilenkin-Krein) Let $W < \mathrm{O}(n)$ be a finite subgroup. A distribtion ξ on $\mathbb{R}^n$ is W-positive-definite if and only if it is the Fourier transform of a positive W-invariant Radon measure μ_ξ, supported on*

$$X_W = \left\{ z \in \mathbb{C}^n : \overline{z} \in W.z \right\} \subset \mathbb{C}^n .$$

We refer to the measure μ_ξ as the *Krein measure* of the distribution ξ. We present Bopp's proof of this theorem using a slightly restricted version of the classical Plancherel-Godement theorem for complex $*$-algebras. There is no guarantee for the measure μ_ξ to be uniquely defined, but if we extend our test function space to the so called *Gelfand-Shilov space* $\mathscr{S}_\alpha(\mathbb{R}^n)$ with parameter $\alpha \geq 0$, then uniqueness can be proved and the support of μ_ξ is restricted to

$$X_{\alpha,W} = \left\{ z \in \mathbb{C}^n : \overline{z} \in W.z \text{ and } \|\mathrm{Im}(z)\| \leq \alpha \right\} \subset \mathbb{C}^n .$$

While this result is interesting in itself, it turns out to be very useful in the context of diffraction on Lie groups.

1.1.3 Diffraction on Symmetric Spaces

Another family of homogeneous metric spaces $X = G/K$ that are of interest to us is when G is a semisimple connected Lie group with finite center and K is a maximal compact subgroup. Then the space X is a *symmetric space*, i.e. a Riemannian manifold with isometric geodesic symmetries. As before we can construct an autocorrelation measure η_ν with respect to some ergodic measure ν on Ω_o. The *diffraction measure* of ν is defined as the spherical Fourier transform $\widehat{\eta}_\nu$ of the autocorrelation measure. One main objective in the theory of diffraction is to compute these diffraction measures, given an ergodic measure ν. We describe here concisely how one can compute the diffraction measure using the inverse Abel transform on X.

With the Iwasawa decomposition $G = ANK$ and Lie algebra $\mathfrak{a}$ of A, Anker showed in [1] that the spherical Fourier transform $\mathscr{S}$ defines a $*$-isomorphism

$$\mathscr{S} : C_c^\infty(G,K) \to \mathrm{PW}(\mathfrak{a}_\mathbb{C}^*)^W ,$$

where $\mathrm{PW}(\mathfrak{a}_\mathbb{C}^*)$ is the Paley-Wiener space of the complexification of $\mathfrak{a}^*$ and $W < \mathrm{O}(\mathfrak{a}^*)$ is a finite subgroup called the *Weyl group* of G. In this setting, the Abel transform of radial/bi-K-invariant functions f on G can be defined by

$$\mathscr{A} f(H) = \mathrm{e}^{\rho(H)} \int_N f(\mathrm{e}^H n)\, dm_N(n) ,$$

and this transform is a $*$-isomorphism from bi-K-invariant test functions on G to W-invariant test functions on $\mathfrak{a}$. The autocorrelation distribution on $\mathfrak{a}$ is defined as $\xi_\nu = \mathscr{A}^{-1}\eta_\nu$. If we denote by $\mathscr{F} : L^1(\mathfrak{a}) \to C_0(\mathfrak{a}^*)$ the Euclidean Fourier transform

$$\mathscr{F}\varphi(\lambda) = \int_{\mathfrak{a}} \varphi(H)\mathrm{e}^{-i\lambda(H)}\, dm_{\mathfrak{a}}(H),$$

then the spherical Fourier transform decomposes as $\mathscr{S} = \mathscr{F}\mathscr{A}$, which is thought of as a kind of *Fourier slice theorem* on X. Using this we can summarize the situation by the diagram of $*$-isomorphisms,

$$\begin{array}{ccc} C_c^\infty(G,K) & \xrightarrow{\qquad\mathscr{S}\qquad} & \mathrm{PW}(\mathfrak{a}_{\mathbb{C}}^*)^W \\ & {\scriptstyle\mathscr{A}}\searrow \quad \nearrow{\scriptstyle\mathscr{F}} & \\ & C_c^\infty(\mathfrak{a})^W & \end{array} .$$

Anker moreover showed that one can extend all maps involved to $*$-isomorphisms

$$\begin{array}{ccc} \mathscr{C}^p(G,K) & \xrightarrow{\qquad\mathscr{S}\qquad} & \mathscr{S}(\mathfrak{a}_\alpha^*)^W \\ & {\scriptstyle\mathscr{A}}\searrow \quad \nearrow{\scriptstyle\mathscr{F}} & \\ & \mathscr{S}_\alpha(\mathfrak{a})^W & \end{array} ,$$

where $\mathscr{C}^p(G,K)$, $p \in (0,2]$, is the *Harish-Chandra L^p-space*, $\alpha = 2/p - 1$ and $\mathscr{S}(\mathfrak{a}_\alpha^*)$ is the Schwartz space of holomorphic functions on the convex closure $\mathfrak{a}_\alpha^* \subset \mathbb{C}^n$ of $\mathfrak{a}^* + i\alpha W.\rho$. Given that the autocorrelation distribution ξ_ν extends to $\mathscr{S}_\alpha(\mathfrak{a})$ there is a unique Krein measure μ_ν on $X_{\alpha,W}$ such that $\xi_\nu = \mathscr{F}\mu_\nu$, and so

$$\mu_\nu = \mathscr{S}^{-1}\mathscr{A}\mathscr{F}\mu_\nu = \mathscr{S}^{-1}\mathscr{A}\xi_\nu = \mathscr{S}^{-1}\eta_\nu = \widehat{\eta}_\nu .$$

Indeed one can show that if p is small enough then the autocorrelation measure η_ν extends to $\mathscr{C}^p(G,K)$ and consequently that ξ_ν extends to $\mathscr{S}_\alpha(\mathfrak{a})$ for $\alpha = 2/p - 1$. This however is an important matter that we leave for future work.

1.2 Organization of the Paper

In section 2 we recall some of the main results from distribution theory on Euclidean space and clarify notations and conventions. In section 3 we survey some foundational results and properties of positive-definite functions and distributions. The Abel transform is introduced in section 4, where we derive the Guinand distribution as the Abel inverse of a Dirac comb. In section 5 we introduce relatively positive-definite distributions and the Gelfand-Shilov space to then formulate and prove theorem 1.1 in section 6. Lastly, we define the Abel transform on the hyperbolic plane in section 7 as a special case of section 1.3 and determine its inverse on Dirac combs.

1.3 Acknowledgements

First and foremost I would like to sincerely thank my supervisor Michael Björklund for introducing me to the theory that motivated the creation of this thesis, as well as the many encouraging and supportive discussions we've had throughout the project. Secondly, I owe a thank you to all of my friends whom have joined me on weekdays and weekends, both for work and other pleasures in life. Without you I would not be where I am today. Lastly, I would like to my family for supporting me throughout my studies.

2 Preliminaries and Notation

In this paper, the central objects of study are topological $*$-algebras and their vector space duals, as well as continuous operators between them. In particular, we will mostly study functions and distributions on Euclidean space with a slightly different flavour to that of ordinary distribution theory on compactly supported smooth functions.

Let V be a vector space over the complex numbers. Recall that V is a *topological vector space* if it is endowed with a topology such that the addition and scalar multiplication are continuous operations. We denote by V^* the dual vector space of V, consisting of continuous linear functionals on V. The vector space V^* can be given a topology, and we specialize to the following family of spaces:

Definition 2.1. *A Hausdorff topological vector space V is a Frechét space if the topology on V is induced by countably many seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ and V is complete with respect to them.*

It is worth noting that this definition is equivalent to V being a locally convex complete metric space with respect to a translation invariant metric, but we will think of a Frechét space in the sense of our definition. The continuous dual V^* can in this case be identified with functionals $\alpha : V \to \mathbb{C}$ satisfying

$$|\alpha(x)| \le C_k \|x\|_k , \quad C_k \ge 0 ,$$

for *some* $k \in \mathbb{N}$. Note that every Banach space clearly is a Frechét space and in this case the dual space can be made into a Banach space using the operator norm

$$\|\alpha\| = \sup_{\|x\|\le 1} |\alpha(x)| .$$

If X is a locally compact separable metric space, the main examples being Euclidean and hyperbolic space, we can associate to it the vector space $C(X)$ of continuous complex-valued functions on X. We will in this paper make use of the following algebraic subspaces of $C(X)$:

- The space $C_b(X)$ of bounded continuous functions, endowed with the topology induced by the norm
$$\|\varphi\|_\infty = \sup_{x\in X} |\varphi(x)| .$$
- The space $C_0(X) \subset C_b(X)$ of continuous functions vanishing at infinity.
- The space $C_c(X)$ of compactly supported continuous functions on X, endowed with the colimit topology over all compact subsets $K \subset X$. This topology corresponds to uniform convergence on compacta and is generated by the seminorms
$$\|\varphi\|_K = \sup_{x\in K} |\varphi(x)| .$$

If X in addition has a smooth structure, we can consider the vector space $C^\infty(X)$ of smooth complex-valued functions on X. For multiindices $q \in \mathbb{N}^n$ and vectors $z \in \mathbb{C}^n$ we write

$$|q| = |q_1| + \ldots + |q_n| , \quad q! = q_1! \ldots q_n! , \quad z^q = z_1^{q_1} \ldots z_n^{q_n}$$

and we have an action of linear differential operators on $C^\infty(X)$ by

$$\partial^q \varphi = \partial_1^{q_1} \ldots \partial_n^{q_n} \varphi .$$

In $C^\infty(X)$, we have the algebraic subspace $C_c^\infty(X)$ of compactly supported smooth functions with the topology induced by the seminorms

$$\|\varphi\|_{K,p} = \max_{|q|\le p} \sup_{x\in K} |\partial^q \varphi(x)|$$

for all $p \in \mathbb{N}$ and multiindices $q \in \mathbb{N}^n$, $n = \dim X$. In the special case of $X = \mathbb{R}^n$ we also consider the

Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of smooth functions that are bounded by the seminorms

$$\|\varphi\|_p = \max_{|q|\le p} \sup_{x\in\mathbb{R}^n} (1+\|x\|^2)^p |\partial^q \varphi(x)|.$$

All of the spaces above are Frechét spaces with topologies induced by the mentioned seminorms. Functionals in the duals of these spaces can be realized as the following:

- If K is compact, $C(K)^*$ is the space of finite Borel measures on K,
- $C_0(X)^*$ is the space of regular countably additive finite Borel measures on X,
- $C_c(X)^*$ is the space of Radon measures on X,
- $C_c^\infty(X)^*$ is the space of distributions on X, and
- $\mathscr{S}(\mathbb{R}^n)^*$ is the space of tempered distributions on $\mathbb{R}^n$.

The last two identifications are simply definitions, but the first three are consequences of the *Riesz representation theorem*, stating that every functional μ equivalently is a measure on X, acting on continuous functions φ by

$$\mu(\varphi) = \int_X \varphi \, d\mu.$$

The space X for us will throughout the paper be a so called *homogeneous space*.

Definition 2.2. *A metric space X is homogeneous if its isometry group $G = \mathrm{Isom}(X)$ acts transitively on X.*

By the orbit-stabilizer theorem we can identify X with G/K with $K < G$ being the stabilizer of an arbitrary point of X. We can thus identify complex-valued functions on X with left-K-invariant functions on G and the Haar measure m_G on G descends to a measure m_X on X. For such an X, all vector spaces mentioned above have the additional structure of topological $*$-algebras with continuous operations and involutions

$$(\varphi\psi^*)(Kg) = \varphi(Kg)\overline{\psi(Kg)}$$

on $C_b(X), C_0(X)$ and

$$(\varphi * \psi^*)(Kg) = \int_X \varphi(Kh^{-1}g)\psi(Kh)\, dm_X(Kh)$$

on $C_c(X), C_c^\infty(X)$ and $\mathscr{S}(\mathbb{R}^n)$. Lastly, we denote the Fourier transform $\mathscr{F}: L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ by

$$\widehat{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(y) \mathrm{e}^{-i\langle x,y\rangle}\, dy.$$

When restricted to the space $C_c^\infty(\mathbb{R}^n)$ then the Paley-Wiener theorem yields a $*$-isomorphism

$$\mathscr{F}: C_c^\infty(\mathbb{R}^n) \longrightarrow \mathrm{PW}(\mathbb{C}^n),$$

where $\mathrm{PW}(\mathbb{C}^n)$ is the space of holomorphic functions h on $\mathbb{C}^n$ bounded by the seminorms

$$\|h\|_{p,r} = \sup_{z\in\mathbb{C}^n} (1+\|z\|)^p \mathrm{e}^{-r\|\mathrm{Im}(z)\|} |h(z)|, \quad p \in \mathbb{N}_0, r > 0.$$

The Fourier transform moreover extends to an isomorphism of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

Regarding notation, the Haar measure on a locally compact group G will be denoted by m_G and as a special case, the Lebesgue on $\mathbb{R}^n$ with respect to a variable x will simply be written dx. Moreover, if G is a group acting on a vector space V we write V^G for the G-invariant vectors of V. In particular, using the notation from the discussion above, $C(X) = C(G)^K$ and similarly for subspaces of $C(X)$.

3 Positive-definite Distributions

Let G be a locally compact second countable group, for example the isometry group of Euclidean space or the real/complex hyperbolic plane. One of the main objects of study in this thesis will be the notion of positive-definite functions and distributions on G, as well as generalizations of such. In this section we introduce positive-definite functions and demonstrate their significance. First, we define positive-definiteness in an algebraic sense.

Definition 3.1. *A function $f : G \to \mathbb{C}$ is positive-definite if for all $z_1,...,z_n \in \mathbb{C}$ and $g_1,...,g_n \in G$,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(g_i^{-1}g_j)z_i\overline{z}_j \geq 0\,.$$

One important property of positive-definite functions on G is that they determine the unitary representation theory of G. To see how, let (π, V) be a unitary representation of G with cyclic vector $v \in V$. Then the associated matrix coefficient

$$f_\pi(g) = \langle v, \pi(g)v\rangle$$

is continuous and positive-definite as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f_\pi(g_i^{-1}g_j)z_i\overline{z}_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle z_i\pi(g_i)v, z_j\pi(g_j)v\rangle = \left\| \sum_{i=1}^{n} z_i\pi(g)v \right\|^2 \geq 0\,.$$

The question now is when a positive-definite function is the matrix coefficient of a unitary representation of G. It turns out that it holds for continuous integrable positive-definite functions, and hence they parametrize the unitary dual $\widehat{G}$ of G.

Theorem 3.2. *Let $f \in L^1(G)$. Then the following are equivalent:*
(i) f has a positive-definite representative.
(ii) $\int_G (\varphi * \varphi^*) f \, dm_G \geq 0$ for all $\varphi \in C_c^\infty(G)$.
(iii) There is, up to isomorphism, a unitary representation $(\pi_f, \mathcal{H}_f)$ of G with a cyclic vector $v \in \mathcal{H}_f$ such that $f(g) = \langle v, \pi_f(g)v\rangle$ a.e..

Proof. See [7, ch. 3.3] ∎

Corollary 3.3. *Let $f \in L^1(G)$ and assume it has a positive-definite representative. Then f has a continuous representative and it satisfies*

$$|f(g)| \leq f(1) \quad \textit{and} \quad f(g^{-1}) = \overline{f(g)}$$

for all $g \in G$.

Proof. Take the representative $f(g) = \langle v, \pi_f(g)v\rangle$ as in theorem 3.2. Then by the Cauchy-Schwarz inequality,

$$|f(g)| = |\langle v, \pi_f(g)v\rangle| \leq \|v\|^2 = f(1)$$

and by the skew symmetry of the inner product,

$$f(g^{-1}) = \langle v, \pi_f(g^{-1})v\rangle = \langle v, \pi_f(g)^* v\rangle = \langle \pi_f(g)v, v\rangle = \overline{\langle v, \pi_f(g)v\rangle} = \overline{f(g)}\,.$$

∎

The connection with the unitary representations of G tells us that the positive-definite functions determine the Fourier theory on G for $L^2(G)$, and when G is abelian this parametrization can be rephrased in terms of finite positive Borel measures on the unitary dual $\widehat{G}$.

Theorem 3.4. *(Bochner) Let G be an abelian group. A function $f \in C(G)$ is positive-definite if and only if it is the Fourier transform of a unique positive finite Borel measure μ_f on $\widehat{G}$, i.e.*

$$f(g) = \int_{\widehat{G}} \chi(g)\, d\mu_f(\chi).$$

One of the objectives of this paper is to demonstrate how one obtains positive-definite functions, measures and distributions in different settings. In light of theorem 3.2 we define positive-definiteness for distributions on $\mathbb{R}^n$ as follows:

Definition 3.5. *A distribution ξ on $\mathbb{R}^n$ is said to be positive-definite if for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\xi(\varphi * \varphi^*) \geq 0.$$

When considering $\mathbb{R}^n$ as an abelian group, we know that Bochner's theorem holds for positive-definite functions. Schwartz extended this theorem to positive-definite distributions on $\mathbb{R}^n$.

Theorem 3.6. *(Bochner-Schwartz) A distribution $\xi \in C_c^\infty(\mathbb{R}^n)^*$ is positive-definite if and only if it is the Fourier transform of a positive tempered Radon measure μ_ξ on $\mathbb{R}^n$, i.e. for every $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\xi(\varphi) = \int_{\mathbb{R}^n} \widehat{\varphi}(t)\, d\mu_\xi(t).$$

We return to generalizations of this theorem in section 5.

We know that positive-definite functions determine an essential part of the representation theory of groups, so a natural question is what role the positive-definite measures and distributions play. While we do not directly answer this question in this paper, some answers can be motivated by the study of *spherical diffraction* and we refer to [3, 4, 5] for more information on this.

4 The Abel Transform on Euclidean Space

We have introduced positive-definite distributions and we would like to find a family of examples of such. The Bochner-Schwartz theorem classifies positive-definite distributions in terms of the Fourier transform of certain measures, and this motivates the study of other transforms of measures. We will focus on the *Abel transform* of radial functions on $\mathbb{R}^n$, $n \geq 2$, which can be derived using the *Radon transform*. It is moreover of particular interest, since it decomposes the Fourier transform in a so called *slice theorem*.

4.1 The Radon and Abel Transform

The classical Radon transform on $\mathbb{R}^n$ is a transformation that takes a function and produces it's average on a given affine hyperplane. An affine hyperplane H in $\mathbb{R}^n$ is an affine subspace of codimension 1, so there is a pair (x,t), $x \in \mathbb{R}^n \setminus \{0\}$, $t \in \mathbb{R}$ such that

$$H = H_{x,t} = \left\{ y \in \mathbb{R}^n : \langle x, y \rangle = t \right\}$$

and we denote the set of affine hyperplanes in $\mathbb{R}^n$ by $\mathbb{H}_{\text{aff}}$. Note that the pairs $(x,t),(\lambda x, \lambda t)$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ yield the same hyperplane, so without loss of generality, $x \in S^{n-1}$ and we have a bijection

$$\mathbb{H}_{\text{aff}} = (S^{n-1} \times \mathbb{R})/\pm.$$

From this equality one can endow $\mathbb{H}_{\text{aff}}$ with a topology using the quotient map $S^{n-1} \times \mathbb{R} \longrightarrow \mathbb{H}_{\text{aff}}$. The Radon transform is the map $R : C_c(\mathbb{R}^n) \longrightarrow C_c(S^{n-1} \times \mathbb{R})^{\text{even}}$ defined by

$$Rf(x,t) = \int_{H_{x,t}} f \, d\sigma_{x,t} = \int_{H_{x,0}} f(tx+y) \, d\sigma_{x,0}(y),$$

where $\sigma_{x,t}$ is the canonical hypersurface measure on $H_{x,t}$. If $f \in C_c(\mathbb{R}^n)$ is radial then we may without loss of generality take $x = e_1$ and y in the orthogonal space of e_1, so that

$$Rf(x,t) = \int_{\mathbb{R}^{n-1}} f(t,y) \, dy.$$

In terms of the Euclidean geometry $\mathbb{R}^n = (\mathrm{O}(n) \ltimes \mathbb{R}^n)/\mathrm{O}(n)$ we define the Abel transform as the Radon transform restricted to radial functions.

Definition 4.1. *The Abel transform is the map* $\mathscr{A} : C_c(\mathbb{R}^n)^{\mathrm{O}(n)} \longrightarrow C_c(\mathbb{R})^{\text{even}}$ *given by*

$$\mathscr{A}f(t) = \int_{\mathbb{R}^{n-1}} f(t,y) \, dy.$$

If $F \in C_c(\mathbb{R}^n)^{\text{even}}$ such that $f(x) = F(\|x\|)$ then

$$\mathscr{A}f(t) = \int_{\mathbb{R}^{n-1}} F\left(\sqrt{t^2 + \|y\|^2}\right) dy. \tag{4.1}$$

Making the substitutions $s = \|y\|^2$ and $r = \sqrt{t^2+s^2}$ we get

$$\mathscr{A}f(t) = \mathrm{vol}(S^{n-2}) \int_0^\infty F\left(\sqrt{t^2+s^2}\right) s^{n-2} \, ds = \mathrm{vol}(S^{n-2}) \int_{|t|}^\infty F(r)(r^2-t^2)^{\frac{n-3}{2}} \, r \, dr.$$

and we will therefore write the Abel transform in terms of F as

$$\mathscr{A}F(t) = \mathrm{vol}(S^{n-2}) \int_{|t|}^\infty F(r)(r^2-t^2)^{\frac{n-3}{2}} \, r \, dr.$$

To emphasize the dependency n, we write $\mathscr{A}_n$ for the Abel transform on $\mathbb{R}^n$. For example, if we take $n = 2$ then we obtain the classical Abel transform

$$\mathscr{A}_2 F(t) = 2 \int_{|t|}^\infty \frac{F(r) r}{\sqrt{r^2 - t^2}} \, dr.$$

We will later on construct distributions using the inverse Abel transform, and this is of interest because it behaves nicely on the space of radial test functions. In fact, when restricted to compactly supported smooth functions then it defines a $*$-isomorphism of algebras. To see this, we make use of a property of the Radon transform.

Theorem 4.2. *(Fourier Slice Theorem) Let* $f \in C_c(\mathbb{R}^n)$ *and denote by* $\mathscr{F}_1 : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ *the* 1*-dimensional Fourier transform. Then for every* $\xi \in S^{n-1}$, $\lambda \geq 0$

$$\widehat{f}(\lambda\xi) = \int_{\mathbb{R}} Rf(\xi,t) \mathrm{e}^{-i\lambda t} \, dt,$$

or more compactly stated, the diagram

$$\begin{array}{ccc} C_c(\mathbb{R}^n) & \xrightarrow{\quad \mathscr{F} \quad} & C_0(\mathbb{R}^n) \\ & {\scriptstyle R} \searrow \qquad \nearrow {\scriptstyle \mathscr{F}_1} & \\ & C_c(S^{n-1} \times \mathbb{R})^{\text{even}} & \end{array}.$$

commutes.

Proof. The Fourier transform of f is

$$\widehat{f}(\lambda\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-i\langle \lambda\xi, x\rangle}\, dx$$

and if we write $x = t\xi + y$ where $t \in \mathbb{R}$ and y is in the orthogonal space $H_{\xi,0}$ of ξ, then $\langle \lambda\xi, x\rangle = \lambda(t\|\xi\|^2 + \langle y, \xi\rangle) = \lambda t$ and

$$\widehat{f}(\lambda\xi) = \int_{\mathbb{R}} \int_{H_{\xi,0}} f(t\xi + y)\mathrm{e}^{-it\lambda}\, d\sigma_{\xi,0}(y) dt = \int_{\mathbb{R}} Rf(\xi, t)\mathrm{e}^{-i\lambda t}\, dt.$$

■

The Abel transform was defined as the Radon transform restricted to radial functions, and since the Fourier transform is an automorphism of Schwartz space we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^n)^{\mathrm{O}(n)} & \xrightarrow{\quad \mathscr{F} \quad} & \mathscr{S}(\mathbb{R})^{\mathrm{even}} \\ & \searrow^{\mathscr{A}} \quad \nearrow_{\mathscr{F}_1} & \\ & \mathscr{S}(\mathbb{R})^{\mathrm{even}} & \end{array}.$$

Note that the Fourier transforms are isomorphisms, so the same holds for the Abel transform.

Corollary 4.3. *The Abel transform is a* $*$*-isomorphism* $\mathscr{A} : \mathscr{S}(\mathbb{R}^n)^{\mathrm{O}(n)} \to \mathscr{S}(\mathbb{R})^{\mathrm{even}}$.

Now that we have an isomorphism, it is of our interest to determine what the inverse might be. To find it, we return to the Radon transform. We define the *dual Radon transform* $R^* : C_c(S^{n-1} \times \mathbb{R})^{\mathrm{even}} \to C_c(\mathbb{R}^n)$ by

$$R^*\varphi(x) = \int_{H \ni x} \varphi = \int_{S^{n-1}} \varphi(y, \langle x, y\rangle)\, dy.$$

When $\varphi \in C_c(\mathbb{R})^{\mathrm{even}} \subset C_c(S^{n-1} \times \mathbb{R})^{\mathrm{even}}$ then we can make use of polar coordinates to write the dual Radon transform of φ as an integral over $\mathbb{R}$,

$$\begin{aligned} R^*\varphi(x) &= \int_{S^{n-1}} \varphi(\langle x, y\rangle)\, dy \\ &= \int_{S^{n-1}} \varphi(\|x\| y_1)\, dy \\ &= \mathrm{vol}(S^{n-2}) \int_0^\pi \varphi(\|x\|\cos(\theta)) \sin^{n-2}(\theta)\, d\theta \\ &= \mathrm{vol}(S^{n-2}) \int_{-1}^1 \varphi(\|x\| t)(1-t^2)^{\frac{n-3}{2}}\, dt. \end{aligned}$$

Next, denote the *Hilbert transform* on $\mathbb{R}$ by

$$H\varphi(t) = \int_{\mathbb{R}} \frac{\varphi(s)}{t-s}\, ds$$

in the Cauchy principal value sense. The following result can be found in [12, thm. 3.8.].

Lemma 4.4. *Let* $\varphi \in C_c^\infty(S^{n-1} \times \mathbb{R})^{\mathrm{even}}$. *The Radon transform restricts to a* $*$*-isomorphism* $C_c^\infty(\mathbb{R}^n) \to C_c^\infty(S^{n-1} \times \mathbb{R})^{\mathrm{even}}$ *and its inverse is given by*

$$R^{-1}\varphi = \begin{cases} c_n R^* H \varphi^{(n-1)}, \text{ if } n \text{ is even} \\ c_n R^* \varphi^{(n-1)}, \text{ if } n \text{ is odd} \end{cases}, \quad c_n = \frac{\Gamma(\frac{1}{2})}{(2\sqrt{\pi})^{n-1}\Gamma(\frac{n}{2})}.$$

It moreover extends to a $*$*-isomorphism of the corresponding Schwartz spaces.*

Let us compute the inverse of $\mathscr{A}$ for $\varphi \in C_c^\infty(\mathbb{R})$ when n is odd. Write $n = 2d+1$, $d \geq 1$. Then we have $(1-t^2)^{(n-3)/2} = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k t^{2k}$, so

$$\int_{-1}^{1} \varphi^{(n-1)}(\|x\| t)(1-t^2)^{\frac{n-3}{2}} dt = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k \int_{-1}^{1} \varphi^{(2d)}(\|x\| t) t^{2k} dt.$$

Using induction, one can show that

$$\int_{-1}^{1} \varphi^{(l)}(\|x\| t) t^m dt = \sum_{j=0}^{m-1} (-1)^j \frac{m!}{(m-j)!} \frac{\varphi^{(l-(j+1))}(\|x\|) - (-1)^j \varphi^{(l-(j+1))}(-\|x\|)}{\|x\|^{j+1}}$$

whenever $l > m$, from which it follows that

$$\mathscr{A}_{2d+1}^{-1} \varphi(x) = c_{2d+1} \mathrm{vol}(S^{2d-1}) \sum_{k=0}^{d-1} \sum_{j=0}^{2k-1} (-1)^{k+j} \binom{d-1}{k} \frac{(2k)!}{(2k-j)!} \frac{1}{\|x\|^{j+1}} \frac{d^{2d-(j+1)}(\varphi + \check{\varphi})}{dt^{2d-(j+1)}}(\|x\|).$$

This means in particular that $\mathscr{A}^{-1}$ is a local operator in odd dimensions. To compute the inverse in even dimensions, we need to know more about the Hilbert transform. It can easily be checked that it is skew-symmetric on $L^2(\mathbb{R})$, i.e. $\langle H\varphi, \psi \rangle = -\langle \varphi, H\psi \rangle$, so to express $\mathscr{A}^{-1}$ in terms of elementary operators, we need to compute the Hilbert transform of $t \mapsto (1-t^2)^{(n-3)/2} \chi_{[-1,1]}(t)$. Write $n = 2d$, $d \geq 1$, and

$$(1-t^2)^{\frac{n-3}{2}} = \sum_{k=0}^{d-1} \binom{d-1}{k} (-1)^k \frac{t^{2k}}{\sqrt{1-t^2}}.$$

The following facts can be found in [2, p. 243-247]: If $f : \mathbb{R} \to \mathbb{R}$ lies in the domain of the Hilbert transform then

$$H(t \mapsto t f(t)) = t H f(t) + \frac{1}{\pi} \int_{\mathbb{R}} f(s) ds$$

and moreover,

$$H\Big(t \mapsto \frac{\chi_{[-1,1]}(t)}{\sqrt{1-t^2}}\Big) = \frac{1}{\sqrt{t^2-1}} (\chi_{(-\infty,-1]}(t) - \chi_{[1,+\infty)}(t)).$$

Thus it follows by iteration that

$$H\Big(t \mapsto \frac{t^{2k}}{\sqrt{1-t^2}} \chi_{[-1,1]}(t)\Big) = \frac{t^{2k}}{\sqrt{t^2-1}} (\chi_{(-\infty,-1]}(t) - \chi_{[1,+\infty)}(t)) + C_k$$

where

$$C_k = \sum_{j=0}^{2k-1} \frac{1}{\pi} \int_{-1}^{1} \frac{s^j}{\sqrt{1-s^2}} ds = 1 + \sqrt{\frac{2}{\pi}} \sum_{\ell=1}^{k-1} \frac{(2\ell-1)!}{2^{2\ell-1} \ell! (\ell-1)!} < +\infty.$$

Finally, we have that

$$H(t \mapsto (1-t^2)^{(n-3)/2} \chi_{[-1,1]}(t)) = (-1)^{d-1} (t^2-1)^{(n-3)/2} (\chi_{(-\infty,-1]}(t) - \chi_{[1,+\infty)}(t))$$

and the Abel inverse is

$$\mathscr{A}_{2d}^{-1} \varphi(x) = (-1)^d c_{2d} \mathrm{vol}(S^{2d-2}) \int_{\mathbb{R} \setminus [-1,1]} \varphi^{(2d-1)}(\|x\| t)(t^2-1)^{\frac{n-3}{2}} \mathrm{sgn}(t) dt.$$

As the Hilbert transform is a non-local operator, the same will hold for the Abel inverse in even dimensions. We will next apply the inverse Abel transform to measures on $\mathbb{R}^n$ and observe that they yield interesting distributions on $\mathbb{R}$, even in the simplest cases.

4.2 Guinand's Distribution

We saw in the previous subsection that the Abel transform defined a $*$-isomorphism $\mathscr{A} : \mathscr{S}(\mathbb{R}^n)^{\mathrm{O}(n)} \to \mathscr{S}(\mathbb{R})^{\mathrm{even}}$, and so it dualizes to an isomorphism $\mathscr{A} : \mathscr{S}(\mathbb{R})^* \to \mathscr{S}(\mathbb{R}^n)^*$ when restricted to even distributions. It is defined on distributions by the dual construction, i.e.

$$\mathscr{A}\xi(f) = \xi(\mathscr{A}f)$$

for $\xi \in \mathscr{S}(\mathbb{R})^*$, $f \in \mathscr{S}(\mathbb{R}^n)$. The inverse $\mathscr{A}^{-1} : \mathscr{S}(\mathbb{R}^n)^* \to \mathscr{S}(\mathbb{R})^*$ is defined in the same manner. Now, let us give an example using the inverse Abel transform. Consider the standard lattice $\mathbb{Z}^n < \mathbb{R}^n$ and the associated Dirac comb $\delta_{\mathbb{Z}^n}$. For any $f \in \mathscr{S}(\mathbb{R}^n)$, it is given by

$$\delta_{\mathbb{Z}^n}(f) = \sum_{k\in\mathbb{Z}^n} f(k).$$

Understanding the Fourier transform of this measure is a foundational result of Fourier analysis. In this subsection, we reparametrize the Fourier transform as $\widehat{f}(x) = \int_{\mathbb{R}^n} f(y)\mathrm{e}^{-2\pi i\langle x,y\rangle}\,dy$ for clarity of the results.

Theorem 4.5. *(Poisson Summation Formula) Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then*

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{l\in\mathbb{Z}^n} \widehat{f}(l).$$

In terms of the Dirac comb, this formula can equivalently be written as $\widehat{\delta}_{\mathbb{Z}^n} = \delta_{\mathbb{Z}^n}$. Moreover, if $f \in \mathscr{S}(\mathbb{R}^n)$ is radial with corresponding function $F \in \mathscr{S}(\mathbb{R})^{\text{even}}$, then

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{m=0}^{\infty} r_n(m)F(\sqrt{m}),$$

where $r_n(m) = |\{k \in \mathbb{Z}^n : \|k\|^2 = m\}|$ counts the number of lattice points on the sphere of squared radius m. If we let $\varphi \in \mathscr{S}(\mathbb{R})^{\text{even}}$ and $f = \mathscr{A}^{-1}\varphi \in \mathscr{S}(\mathbb{R}^n)^{\mathrm{O}(n)}$, then the Fourier slice theorem says that $\widehat{f}(x) = \widehat{\varphi}(\|x\|)$ and by the Poisson summation formula

$$\sum_{k\in\mathbb{Z}^n} \mathscr{A}^{-1}\varphi(k) = \sum_{k\in\mathbb{Z}^n} f(k) = \sum_{l\in\mathbb{Z}^n} \widehat{f}(l) = \sum_{l\in\mathbb{Z}^n} \widehat{\varphi}(\|l\|) = \sum_{m=0}^{\infty} r_n(m)\widehat{\varphi}(\sqrt{m}).$$

In the language of distributions, the statement is that the measure

$$\mu_n(\varphi) = \sum_{m=0}^{\infty} r_n(m)(\varphi(\sqrt{m}) + \varphi(-\sqrt{m})) \tag{4.2}$$

is the Fourier transform of the distribution $\xi_n = \mathscr{A}^{-1}\delta_{\mathbb{Z}^n}$ on $\mathbb{R}$. With the formulas for the inverse Abel transform from section 3.1 we can write out the formula $\xi_n = \widehat{\mu}_n$ in terms of test functions. This is of particular interest when n is odd, since the locality of $\mathscr{A}^{-1}$ then preserves discrete support of distributions. In the simplest case, when $n = 3$, we have that $c_3 = (2\pi)^{-1} = \mathrm{vol}(S^1)^{-1}$ and $\mathscr{A}^{-1}\varphi(x) = (\varphi'(\|x\|) - \varphi'(-\|x\|))/\|x\|$. As $x \to 0$ then $\mathscr{A}^{-1}\varphi(0) = -2\varphi''(0)$ and so $\xi_3 = \widehat{\mu}_3$ can be written as

$$-2\varphi''(0) + \sum_{m=1}^{\infty} \frac{r_3(m)}{\sqrt{m}}(\varphi'(\sqrt{m}) - \varphi'(-\sqrt{m})) = \sum_{m=0}^{\infty} r_3(m)(\widehat{\varphi}(\sqrt{m}) + \widehat{\varphi}(-\sqrt{m})).$$

Guinand introduced in [11] the distribution

$$\sigma_3(\varphi) = -2\varphi'(0) + \sum_{m=1}^{\infty} \frac{r_3(m)}{\sqrt{m}}(\varphi(\sqrt{m}) - \varphi(-\sqrt{m}))$$

and it is clear that $\xi_3 = -\sigma_3'$. If we define $\psi(t) = \widehat{\varphi}(t)/(it)$, then $\psi(0) = -2\widehat{\varphi}'(0)/i$ and

$$\sigma_3(\varphi) = \mu_3(\psi) = -\frac{2}{i}\widehat{\varphi}'(0) + \sum_{m=1}^{\infty} r_3(m)\Big(\frac{\widehat{\varphi}(\sqrt{m})}{i\sqrt{m}} - \frac{\widehat{\varphi}(-\sqrt{m})}{i\sqrt{m}}\Big) = \frac{1}{i}\sigma_3(\widehat{\varphi}) = -i\sigma_3(\widehat{\varphi}),$$

so we have a non-trivial summation formula that is compactly written as

$$\widehat{\sigma}_3 = -i\sigma_3 .$$

For the more general formula $\xi_n = \widehat{\mu}_n$ for $n = 2d+1$ odd, we can write it out as

$$c_{2d+1}\mathrm{vol}(S^{2d-1})\sum_{m=0}^{\infty} r_{2d+1}(m)\sum_{k=0}^{d-1}\sum_{j=0}^{2k-1}(-1)^{k+j}\binom{d-1}{k}\frac{(2k)!}{(2k-j)!}\frac{1}{m^{(j+1)/2}}\frac{d^{2d-(j+1)}(\varphi+\check{\varphi})}{dt^{2d-(j+1)}}(\sqrt{m})$$
$$= \sum_{m=0}^{\infty} r_{2d+1}(m)(\widehat{\varphi}(\sqrt{m})+\widehat{\varphi}(-\sqrt{m})).$$

In the case where $n = 2d$ is even, we have

$$(-1)^d c_{2d}\mathrm{vol}(S^{2d-2})\sum_{m=0}^{\infty} r_{2d}(m)\int_{\mathbb{R}\setminus[-1,1]}\varphi^{(2d-1)}(\sqrt{m}t)(t^2-1)^{\frac{n-3}{2}}\mathrm{sgn}(t)\,dt = \sum_{m=0}^{\infty} r_{2d}(m)(\widehat{\varphi}(\sqrt{m})+\widehat{\varphi}(-\sqrt{m})),$$

but note that since $\mathscr{A}^{-1}$ is non-local the distribution ξ_{2d} defined by the left hand side does not necessarily have discrete support. We can moreover estimate the number of lattice points in a ball of radius m by $2^n m^n$, so

$$|\mu_n(\varphi)| \leq 2^n \sum_{m=0}^{\infty} m^n(|\varphi(\sqrt{m})| - |\varphi(-\sqrt{m})|) \leq 2^{n+1}\left\|\varphi\right\|_p \sum_{m=0}^{\infty}\frac{m^n}{(1+m)^p} < +\infty$$

whenever $p > n+1$. Thus μ_n is a tempered measure and ξ_n is a positive-definite distribution by the Bochner-Schwartz theorem. Note that we only knew that ξ_n was positive-definite with respect to *even* test functions. The rest of this paper will be dedicated to distributions that are positive-definite with respect to a strict subspace of test functions.

5 Relatively Positive-Definite Distributions

5.1 Krein's Theorem

We saw that the Abel transform on $\mathbb{R}^n$ defines a linear $*$-isomorphism

$$\mathscr{A} : \mathscr{S}(\mathbb{R}^n)^{\mathrm{O}(n)} \longrightarrow \mathscr{S}(\mathbb{R})^{\mathrm{even}}$$

and so radial positive-definite measures η on $\mathbb{R}^n$ induces distributions $\xi = \mathscr{A}^{-1}\eta$ that are positive-definite with respect to even functions on $\mathbb{R}$. In the case of the Dirac comb $\eta = \delta_{\mathbb{Z}^n}$ this turned out to be a positive-definite distribution, but the notion of purely *evenly positive-definite* functions/distributions exists. Consider the basic example $f(x) = \cosh(tx)$ on $\mathbb{R}$, $t \in \mathbb{R}$. It satisfies

$$\int_{\mathbb{R}}(\varphi * \varphi^*)(x)f(x)\,dx = \widehat{\varphi * \varphi^*}(it) = \widehat{\varphi}(it)\overline{\widehat{\varphi}(-it)} = |\widehat{\varphi}(it)|^2 \geq 0$$

for all even $\varphi \in C_c^\infty(\mathbb{R})$, so f is evenly positive-definite. However, $f(x) \geq f(0) = 1$ for all $x \in \mathbb{R}$, so f cannot possibly be positive-definite by corollary 3.3. This notion of *relative positive-definiteness* can on Euclidean space be generalized as the following:

Definition 5.1. *Let $W < \mathrm{GL}_n(\mathbb{R})$. A distribution ξ on $\mathbb{R}^n$ is positive-definite relative to W, or in short W-positive-definite, if for any W-invariant $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\xi(\varphi * \varphi^*) \geq 0 .$$

This definition captures a wide range of distributions, including the positive-definite distributions when $W = \{0\}$. First, let us consider the simplest non-trivial example, when $n = 1$ and $W = \mathrm{O}(1)$. The first classification of relatively positive-definite functions was due to M.G. Krein.

Theorem 5.2. *(Krein) A continuous function f on $\mathbb{R}$ is evenly positive-definite if and only if there is a positive finite Borel measure μ^+ on $\mathbb{R}$ and a positive Radon measure μ^- on $\mathbb{R}$ such that*

$$f(x) = \int_{\mathbb{R}} \cos(tx)\, d\mu^+(t) + \int_{\mathbb{R}} \cosh(tx)\, d\mu^-(t).$$

Note that the pair of measures (μ^+, μ^-) can be interpreted as a measure μ_f supported on the cross $\mathbb{R} \cup i\mathbb{R} \subset \mathbb{C}$, so that

$$f(x) = \int_{\mathbb{R}\cup i\mathbb{R}} \cos(zx)\, d\mu_f(z),$$

i.e. f is the complex Fourier transform of μ_f. We will call the measure $\mu_f = (\mu^-, \mu^+)$ and its generalizations the *Krein measure* of f. A simple example of a Krein measure would be that of $f(x) = \cosh(tx)$ from earlier, in which case we can take $\mu_f = \delta_{it}$. A non-trivial example of this is for example $f(x) = \mathrm{e}^{x^2}$. It is not a positive-definite function as it does not attain its maximum at 0. However it is evenly positive-definite by Krein's theorem, for if we take $\mu^+(t) = 0$, $\mu^-(t) = \mathrm{e}^{-t^2} m_{\mathbb{R}}(t)$ and μ the corresponding measure on $\mathbb{R} \cup i\mathbb{R}$ then

$$\int_{\mathbb{C}} \cos(zx)\, d\mu(z) = \int_{\mathbb{R}} \mathrm{e}^{tx}\, d\mu^-(t) = \mathrm{e}^{-t^2}\Big|_{t=ix} = f(x).$$

Also note that φ defines a non-tempered distribution, so we have given an example of a evenly positive-definite function on $\mathbb{R}^n$ that is not positive-definite. Gelfand and Vilenkin extended the theorem of Krein to higher dimensions and distributions on such spaces. Consider the reflection group $\mathrm{O}(1)^n < \mathrm{O}(n)$ acting on $\mathbb{R}^n$. We say that a distribution is *evenly positive-definite* if it is $\mathrm{O}(1)^n$-positive-definite.

Theorem 5.3. *(Gelfand-Vilenkin-Krein) A distribution ξ on $\mathbb{R}^n$ is evenly positive-definite if and only if it is the Fourier transform of a positive Radon measure μ_ξ supported on $(\mathbb{R}\cup i\mathbb{R})^n \subset \mathbb{C}^n$. That is,*

$$\xi(\varphi) = \int_{(\mathbb{R}\cup i\mathbb{R})^n} \widehat{\varphi}(z)\, d\mu_\xi(z)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Note that there has been no mention of the uniqueness of the measure μ_ξ for a given distribution ξ. We next give an example of non-uniqueness, which also can be found in [10, ch. 6.4.].

5.2 An Example of Non-uniqueness

An even measure on $\mathbb{C}$ defines a functional on $\mathrm{PW}(\mathbb{C})^{\mathrm{even}}$, so we will construct two different measures μ and ν on $\Omega^+ = \mathbb{R}^+ \cup i\mathbb{R}^+ \subset \mathbb{C}$ such that

$$\int_{\mathbb{C}} h\, d\mu = \int_{\mathbb{C}} h\, d\nu$$

for every even $h \in \mathrm{PW}(\mathbb{C})$. To do this, we construct a non-trivial function f on the first quadrant of $\mathbb{C}$ satisfying

$$\int_0^\infty h(x) f(x)\, dx = i \int_0^\infty h(iy) f(iy)\, dy$$

for every h. Then if we write $f = u + iv$ for real valued functions u, v on $\mathbb{C}$ we get that the equality above is equivalent to

$$\int_0^\infty h(x)u(x)\, dx + \int_0^\infty h(iy)v(iy)\, dx = i\Big(-\int_0^\infty h(x)v(x)\, dx + \int_0^\infty h(iy)u(iy)\, dy\Big).$$

If h is real-valued or purely imaginary-valued, then both sides of this equation must be zero, and by linearity we must have that both sides are zero for any $h \in \mathrm{PW}(\mathbb{C})^{\text{even}}$. Thus

$$\int_0^\infty h(x)u(x)\,dx + \int_0^\infty h(iy)v(iy)\,dx = 0$$

and if we decompose u, v into non-negative functions $u = u^+ - u^-, v = v^+ - v^-$ then this can be written as

$$\int_0^\infty h(x)u^+(x)\,dx + \int_0^\infty h(iy)v^-(iy)\,dy = \int_0^\infty h(x)u^-(x)\,dx + \int_0^\infty h(iy)v^+(iy)\,dy.$$

If we define measures

$$d\mu(x+iy) = u^+(x)\,dx + v^-(iy)\,dy \quad \text{and} \quad d\nu(x+iy) = u^-(x)\,dx + v^+(iy)\,dy$$

on Ω^+ then $\mu \neq \nu$ as $u^+ \neq u^-$ and $v^+ \neq v^-$, and it is clear that

$$\int_{\mathbb{C}} h(z)\,d\mu(z) = \int_{\mathbb{C}} h(z)\,d\nu(z)$$

for all $h \in \mathrm{PW}(\mathbb{C})^{\text{even}}$. Now for the construction of f.

We construct a holomorphic function f on the interior of the positive quadrant of $\mathbb{C}$, which

1. extends to a continuous function on $\Omega^+ \setminus \{0\}$, and
2. satisfies
$$\lim_{R\to+\infty} \int_{\gamma_R^+} \mathrm{e}^{ty} |f(x+iy)|\,dm_R^+(x,y) = 0$$
for all $t > 0$, where m_R^+ for each $R > 0$ is the arcwise measure on γ_R^+, induced from the Euclidean metric on $\mathbb{C}^n$.

Here γ_R^+ denotes the quarter circle of radius R, oriented counterclockwise. More generally, for $0 < r < R$ we denote by γ_R^- the same curve with clockwise orientation and $\gamma_{R,r}^\pm$ the boundary of the quarter annulus with outer and inner radii R and r, oriented counterclockwise/clockwise. By Cauchy's theorem,

$$0 = \int_{\gamma_{R,r}^+} h(z)f(z)\,dz = \int_r^R h(x)f(x)\,dx + \int_{\gamma_R^+} h(z)f(z)\,dz - \int_r^R h(iy)f(iy)\,i\,dy + \int_{\gamma_r^-} h(z)f(z)\,dz.$$

Since $h \in \mathrm{PW}(\mathbb{C})$, there are constants $C, t \geq 0$ dependent on h such that $|h(z)| \leq C\mathrm{e}^{t|\mathrm{Im}(z)|}$, and so

$$\left|\int_{\gamma_R^+} h(z)f(z)\,dz\right| \leq C \int_{\gamma_R^+} \mathrm{e}^{ty}|f(x+iy)|\,dm_R^+(x,y) \longrightarrow 0$$

as $R \to +\infty$. Also, as f extends to a continuous and in particular bounded function near 0 then

$$\left|\int_{\gamma_r^-} h(z)f(z)\,dz\right| \leq \sup_{0<|z|\leq 1} |h(z)f(z)|\,\ell(\gamma_r^-) \longrightarrow 0$$

as $r \to 0$, where $\ell(\gamma)$ denotes the length of a curve $\gamma \subset \mathbb{C}$. This means that

$$\int_0^\infty h(x)f(x)\,dx = i\int_0^\infty h(iy)f(iy)\,dy.$$

Now, let $a \in (1,2)$, $b = a\pi/4$ and consider the function

$$f_{a,b}(z) = \mathrm{e}^{-z^a \mathrm{e}^{-ib}} = \mathrm{e}^{-|z|^a \mathrm{e}^{i(a\arg(z)-b)}}.$$

This function is by the choice of a holomorphic on the interior of the first quadrant and it extends to a continuous function on the first quadrant. We have the bound

$$|\mathrm{e}^{-z^a\mathrm{e}^{-ib}}| \le \mathrm{e}^{-|z|^a\cos(a(\arg(z)-\pi/4))}$$

for all z in the first quadrant, and since $0 < \arg(z) < \pi/2$, $c_a = \inf_z \cos(a(\arg(z)-\pi/4)) = \cos(a\pi/4) > 0$ for our choice of a. Thus

$$\int_{\gamma_R^+} \mathrm{e}^{ty}|f_{a,b}(x+iy)|\,dm_R^+(x,y) \le \mathrm{e}^{tR-c_aR^a}\,\ell(\gamma_R^-) = \mathrm{e}^{R(t-c_aR^{a-1})}\frac{\pi R}{2} \longrightarrow 0$$

as $R \to +\infty$. Taking $f = f_{a,b}$ we have provided an example as required.

To obtain uniqueness of the Krein measure, we will need asymptotic bounds on the distributions in question. We get such bounds by allowing for a broader family of test functions, and we will extend them to a space studied by Gelfand and Shilov in [9].

5.3 The Gelfand-Shilov Space $\mathscr{S}_\alpha(\mathbb{R}^n)$

To obtain uniqueness of Krein measures we need to put some restrictions on the asymptotics of the distribution ξ. Gelfand and Shilov introduced the Frechét space $\mathscr{S}_\alpha(\mathbb{R}^n)$, $\alpha \ge 0$, of smooth functions on $\mathbb{R}^n$ bounded by the seminorms

$$\|\varphi\|_{\alpha,p} = \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)|\partial^q\varphi(x)|\,, \quad M_p(x) = (1+\|x\|^2)^p\mathrm{e}^{\alpha\|x\|}\,.$$

The topology on this space is induced by convergence in these seminorms and note that $\mathscr{S}_0(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)$ is the ordinary Schwartz space on $\mathbb{R}^n$. Moreover, it is easy to see that there is a canonical injection $\mathscr{S}_\alpha(\mathbb{R}^n) \to \mathscr{S}_\beta(\mathbb{R}^n)$ whenever $\alpha \ge \beta$ and so all such function spaces embed into the space of Schwartz functions. It is also clear that $\mathscr{S}_\alpha(\mathbb{R}^n)$ contains all compactly supported smooth functions on $\mathbb{R}^n$.

Lemma 5.4. *The canonical map*

$$C_c^\infty(\mathbb{R}^n) \longrightarrow \mathscr{S}_\alpha(\mathbb{R}^n)$$

is a continuous injection, whose image is a dense subspace of $\mathscr{S}_\alpha(\mathbb{R}^n)$.

Proof. If $\varphi \in C_c^\infty(\mathbb{R}^n)$ then for any compact $K \subset \mathbb{R}^n$ containing the support of φ,

$$\|\varphi\|_{\alpha,p} = \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)|\partial^q\varphi(x)| \le \sup_{x\in K} M_p(x) \sum_{|q|\le p}\sup_{x\in K}|\partial^q\varphi(x)| < +\infty,$$

so $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$. Continuity of the map also follows from this bound and if $\varphi = 0$ in $\mathscr{S}_\alpha(\mathbb{R}^n)$ then $\|\varphi\|_\infty \le \|\varphi\|_{\alpha,0} = 0$, meaning that $\varphi = 0$ in $C_c^\infty(\mathbb{R}^n)$ and proving injectivity. It remains to show that we can approximate $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ by compactly supported smooth functions.

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a bump function taking the value 1 on the unit ball in $\mathbb{R}^n$, and consider for each $m \in \mathbb{N}$ the function $\chi_m(x) = \chi(x/m)$. If $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ then $\chi_m\varphi \in C_c^\infty(\mathbb{R}^n)$ and we claim that $\chi_m\varphi \to \varphi$ in $\mathscr{S}_\alpha(\mathbb{R}^n)$. To see this, note that for every multiindex q,

$$\partial^q\chi_m\varphi = \sum_{r\le q} m^{|r|-|q|}\binom{|q|}{|r|}\partial^{q-r}\chi_m\partial^r\varphi = \sum_{r<q} m^{|r|-|q|}\binom{|q|}{|r|}\partial^{q-r}\chi_m\partial^r\varphi + \chi_m\partial^q\varphi\,,$$

so

$$\begin{aligned}
\left\|\chi_m\varphi-\varphi\right\|_{\alpha,p} &= \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)|\partial^q(\chi_m\varphi-\varphi)(x)| \\
&= \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)\Big|\sum_{r\le q} m^{|r|-|q|}\binom{|q|}{|r|}\partial^{q-r}\chi_m(x)\partial^r\varphi(x)-\partial^q\varphi(x)\Big| \\
&\le \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)|\chi_m(x)\partial^q\varphi(x)-\partial^q\varphi(x)| \\
&\quad +\max_{|q|\le p}\sum_{r<q} m^{|r|-|q|}\binom{|q|}{|r|}\sup_{x\in\mathbb{R}^n} M_p(x)|\partial^{q-r}\chi_m(x)\partial^r\varphi(x)| \\
&\le \max_{|q|\le p}\sup_{x\in\mathbb{R}^n} M_p(x)|(1-\chi_m(x))||\partial^q\varphi(x)| \\
&\quad + \max_{|r|\le|q|\le p}\frac{|q|!\#\{r<q\}}{m}\left\|\partial^{q-r}\chi_m\right\|_\infty \sup_{x\in\mathbb{R}^n} M_p(x)|\partial^r\varphi(x)| \\
&=: \max_{|q|\le p} A_q(m)+\max_{|r|\le|q|\le p} B_{q,r}(m).
\end{aligned}$$

It suffices to show $A_q(m), B_{q,r}(m)\to 0$ as $m\to+\infty$. Note that $1-\chi(x/m)\le 1-\chi_{B(0,m)}(x)\le \frac{1}{m}(1+\|x\|^2)$, so $M_p(x)(1-\chi_m(x))\le m^{-1}M_{p+1}(x)$ and

$$A_q(m)\le \frac{1}{m}\sup_{x\in\mathbb{R}^n} M_{p+1}(x)|\partial^q\varphi(x)|\le \frac{\left\|\varphi\right\|_{\alpha,p+1}}{m}\longrightarrow 0$$

as $m\to+\infty$. Moreover, $|\partial^{q-r}\chi_m(x)|=m^{-(|q|-|r|)}|\partial^{q-r}\chi(x/m)|\le m^{-(|q|-|r|)}\left\|\partial^{q-r}\chi\right\|_\infty$, so

$$B_{q,r}(m)\le \frac{|q|!\#\{r<q\}}{m^{1+|q|-|r|}}\left\|\partial^{q-r}\chi\right\|_\infty\left\|\varphi\right\|_{\alpha,p}\longrightarrow 0$$

as $m\to+\infty$. ■

The space $\mathscr{S}_\alpha(\mathbb{R}^n)$ turns out to have a natural structure of a $*$-algebra when endowed with the operation of convolution

$$(\varphi*\psi)(x)=\int_{\mathbb{R}^n}\varphi(x-y)\psi(y)\,dy$$

and the involution

$$\varphi^*(x)=\overline{\varphi(-x)}.$$

By making the substitution $x\mapsto -x$ we see that $\left\|\varphi^*\right\|_{\alpha,p}=\left\|\varphi\right\|_{\alpha,p}$ as M_p is even in x. This means that the involution is continuous and we would like to say the same about the convolution.

Proposition 5.5. *Let $\varphi,\psi\in\mathscr{S}_\alpha(\mathbb{R}^n)$. Then the inequality*

$$\left\|\varphi*\psi\right\|_{\alpha,p}\le 2^p\left\|\varphi\right\|_{\alpha,p}\left\|\psi\right\|_{\alpha,p}$$

holds. In particular, the convolution on $\mathscr{S}_\alpha(\mathbb{R}^n)$ is a continuous operation.

Proof. The parallellogram law

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2)$$

implies that

$$1+\|x+y\|^2\le 1+2(\|x\|^2+\|y\|^2)\le 2(1+\|x\|^2+\|y\|^2)\le 2(1+\|x\|^2)(1+\|y\|^2),$$

which with the triangle inequality yield the estimate

$$M_p(x+y)=(1+\|x+y\|^2)^p\mathrm{e}^{\alpha\|x+y\|}\le 2^p(1+\|x\|^2)^p(1+\|y\|^2)^p\mathrm{e}^{\alpha(\|x\|+\|y\|)}=2^pM_p(x)M_p(y)$$

From this we get that for all multiindices q such that $|q| \le p$

$$
\begin{aligned}
M_p(x)|\partial^q(\varphi * \psi^*)(x)| &\le M_p(x) \sup_{y\in\mathbb{R}^n} |\partial^q \varphi(x-y)||\psi(y)| \\
&\le 2^p \sup_{y\in\mathbb{R}^n} M_p(x-y)|\partial^q\varphi(x-y)|M_p(y)|\psi(y)| \\
&\le 2^p \max_{|r|\le p} \sup_{y\in\mathbb{R}^n} M_p(x-y)|\partial^r\varphi(x-y)| \max_{|s|\le p} \sup_{y\in\mathbb{R}^n} M_p(y)|\partial^s\psi(y)| \\
&= 2^p \left\|\varphi\right\|_{\alpha,p} \left\|\psi\right\|_{\alpha,p} .
\end{aligned}
$$

■

We are interested in studying distributions as continuous functionals on $\mathscr{S}_\alpha(\mathbb{R}^n)$, and it will be useful to convert such distributions into smooth functions by convolution with a test function. If $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ and $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ then we define their convolution to be the function

$$
(\xi * \varphi)(x) = \xi(\tau_x \varphi),
$$

where $\tau_x\varphi(y) = \varphi(x-y)$.

Lemma 5.6. *Let $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ and $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$. Then*
(i) $\xi * \varphi \in C(\mathbb{R}^n)$,
(ii) $\xi * \varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$, and
(iii) $\xi(\varphi * \psi) = \int_{\mathbb{R}^n} \psi(x)(\xi * \check{\varphi})(x)\, dx$.

Proof. *(i)* Note that for every integer $p \ge 0$ there is a constant $C_{\xi,p}$ such that

$$
\begin{aligned}
|(\xi * \varphi)(x)| &\le C_{\xi,p} \left\|\tau_x\varphi\right\|_{\alpha,p} \\
&= C_{\xi,p} \max_{|q|\le p} \sup_{y\in\mathbb{R}^n} M_p(y)|\partial^q\varphi(x-y)| \\
&\le C_{\xi,p} 2^p M_p(x) \sup_{y\in\mathbb{R}^n} M_p(y-x)|\partial^q\varphi(x-y)| \\
&= C_{\xi,p} 2^p M_p(x) \left\|\varphi\right\|_{\alpha,p} .
\end{aligned}
\tag{5.1}
$$

This means in particular that

$$
|(\xi * \varphi)(x+h) - (\xi * \varphi)(x)| = |\xi(\tau_x(\tau_h\varphi - \varphi))| \le C_{\xi,0} \mathrm{e}^{\alpha\|x\|} \left\|\tau_h\varphi - \varphi\right\|_{\alpha,0} .
$$

The multivariate mean value theorem tells us that

$$
|\varphi(x+h) - \varphi(x)| \le \|h\| \left\|\nabla\varphi(x+\theta_h)\right\| \le \|h\| \max_{1\le i\le n} |\partial_i\varphi(x+\theta_h)|
$$

where $\theta_h \in B(0, \|h\|)$, so by the triangle inequality,

$$
\begin{aligned}
\left\|\tau_h\varphi - \varphi\right\|_{\alpha,0} &= \sup_{y\in\mathbb{R}^n} |\varphi(y+h) - \varphi(y)| \mathrm{e}^{\alpha\|y\|} \\
&\le \sup_{y\in\mathbb{R}^n} \|h\| \max_{1\le i\le n} |\partial_i\varphi(y+\theta_h)| \mathrm{e}^{\alpha\|y+\theta_h\|} \mathrm{e}^{\alpha\|\theta_h\|} \\
&\le \|h\| \left\|\varphi\right\|_{\alpha,1} \mathrm{e}^{\alpha\|\theta_h\|} \longrightarrow 0
\end{aligned}
$$

as $h \to 0$.

(ii) As $\xi * \varphi$ is continuous, it is locally integrable and by *(i)* we have that

$$\left|\int_{\mathbb{R}^n} \psi(x)(\xi * \varphi)(x)\,dx\right| \le C_{\xi,p} 2^p \|\varphi\|_{\alpha,p} \int_{\mathbb{R}^n} \psi(x) M_p(x)\,dx$$
$$\le C_{\xi,p} 2^p \|\varphi\|_{\alpha,p} \|\psi\|_{\alpha,p+k} \int_{\mathbb{R}^n} \frac{M_p(x)}{M_{p+k}}\,dx.$$

The integral

$$\int_{\mathbb{R}^n} \frac{M_p(x)}{M_{p+k}}\,dx = \int_{\mathbb{R}^n} \frac{dx}{(1+\|x\|^2)^k} = \mathrm{vol}(S^{n-1}) \int_0^\infty \frac{r^{n-1}\,dr}{(1+r^2)^k}$$

converges if and only if $2k \ge n$, and if we pick such a k then we are done.

(iii) Restricting ξ to a distribution on test functions in $C_c^\infty(\mathbb{R}^n)$, standard distribution theory tells us that $\xi(\varphi * \psi) = \langle \xi * \check{\varphi}, \psi\rangle$ for all $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. Moreover, since $C_c^\infty(\mathbb{R}^n)$ canonically injects onto a dense subspace of $\mathscr{S}_\alpha(\mathbb{R}^n)$ by lemma 5.4 then we can for each $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ pick a sequence $\psi_m \in C_c^\infty(\mathbb{R}^n)$ such that $\psi_m \to \psi$ in $\mathscr{S}_\alpha(\mathbb{R}^n)$. Since ξ and $\xi * \varphi$ define continuous functionals on $\mathscr{S}_\alpha(\mathbb{R}^n)$ then

$$\xi(\varphi * \psi_m) \longrightarrow \xi(\varphi * \psi) \quad \text{and} \quad \langle \xi * \check{\varphi}, \psi_m\rangle \longrightarrow \langle \xi * \check{\varphi}, \psi\rangle$$

as $m \to +\infty$ for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. By uniqueness of the limit, $\xi(\varphi * \psi) = \langle \xi * \check{\varphi}, \psi\rangle$ for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)$. Similarly, if we take $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ and a sequence $\varphi_m \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi_m \to \varphi$ in $\mathscr{S}_\alpha(\mathbb{R}^n)$, then $\xi(\varphi_m * \psi) \to \xi(\varphi * \psi)$ for all $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)$, so it remains to show that $\langle \xi * \check{\varphi}_m, \psi\rangle \to \langle \xi * \check{\varphi}, \psi\rangle$. The estimate made in *(ii)* tells us that for $2k \ge n$,

$$\left|\int_{\mathbb{R}^n} \psi(x)(\xi * (\check{\varphi}_m - \check{\varphi}))(x)\,dx\right| \le C \|\varphi_m - \varphi\|_{\alpha,p} \longrightarrow 0,$$

where

$$C = C_{\xi,p} 2^p \|\psi\|_{\alpha,p+k} \mathrm{vol}(S^{n-1}) \int_0^\infty \frac{r^{n-1}\,dr}{(1+r^2)^k}.$$

Finally we have that

$$\xi(\varphi * \psi) = \langle \xi * \check{\varphi}, \psi\rangle = \int_{\mathbb{R}^n} \psi(x)(\xi * \check{\varphi})(x)\,dx$$

for all $\varphi, \psi \in \mathscr{S}_\alpha(\mathbb{R}^n)$.

■

Corollary 5.7. *If $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ and $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ then $\xi * \varphi \in C^\infty(\mathbb{R}^n)$ and*

$$\partial^q(\xi * \varphi) = \xi * \partial^q \varphi = \partial^q \xi * \varphi$$

for all multiindices q.

Proof. Let $\varphi, \psi \in \mathscr{S}_\alpha(\mathbb{R}^n)$. By properties *(ii)* and *(iii)* of lemma 5.6 we have that

$$\langle \partial^q(\xi * \varphi), \psi\rangle = (-1)^{|q|}\langle \xi * \varphi, \partial^q \psi\rangle = (-1)^{|q|}\langle \xi, \check{\varphi} * \partial^q \psi\rangle = (-1)^{|q|}\langle \xi, \partial^q \check{\varphi} * \psi\rangle.$$

First note that

$$(-1)^{|q|}\langle \xi, \partial^q \check{\varphi} * \psi\rangle = \langle \xi, (\partial^q \varphi)^\vee * \psi\rangle = \langle \xi * \partial^q \varphi, \psi\rangle,$$

and secondly,

$$(-1)^{|q|}\langle \xi, \partial^q \check{\varphi} * \psi\rangle = (-1)^{|q|}\langle \xi, \partial^q(\check{\varphi} * \psi)\rangle = \langle \partial^q \xi, \check{\varphi} * \psi\rangle = \langle \partial^q \xi * \varphi, \psi\rangle.$$

As $\mathscr{S}_\alpha(\mathbb{R}^n)$ separates points in $\mathbb{R}^n$ then $\partial^q(\xi * \varphi) = \xi * \partial^q \varphi = \partial^q \xi * \varphi$ as functions. ■

In converse to the above lemma, we would like to know when a function defines a continuous functional on $\mathscr{S}_\alpha(\mathbb{R}^n)$. Using the seminorms $\|\cdot\|_{\alpha,p}$, we can find a criterion for just that.

Lemma 5.8. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $f \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ if and only if*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+\|x\|^2)^p \mathrm{e}^{\alpha\|x\|}}\,dx < +\infty.$$

for every $p \in \mathbb{N}_0$.

Proof. If $\int_{\mathbb{R}^n} |f(x)| M_p(x)^{-1}\,dx < +\infty$ for some $p \in \mathbb{N}$ then for every φ in the dense subspace $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}_\alpha(\mathbb{R}^n)$ we have

$$\Big|\int_{\mathbb{R}^n} \varphi(x) f(x)\,dx\Big| \le \int_{\mathbb{R}^n} |\varphi(x)||f(x)|\,dx \le \|\varphi\|_{\alpha,p} \int_{\mathbb{R}^n} \frac{|f(x)|}{M_p(x)}\,dx < +\infty.$$

This means precisely that $f \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$.

Conversely, suppose $f \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$. Then by continuity of the functional associated to f there is for every $p \in \mathbb{N}_0$ a constant $C_p \ge 0$ such that $|\langle f, \varphi\rangle| \le C_p \|\varphi\|_{\alpha,p}$ for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. Now let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a bump function that is 1 on the unit ball and consider the functions

$$\varphi_m(x) = \frac{\chi_m(x)}{M_p(x)},$$

where $\chi_m(x) = \chi(x/m)$. As $|\partial^q \chi_m(x)| \le m^{-|q|} \|\partial^q \chi\|_\infty$ for all multiindices q then $\|\varphi_m\|_{\alpha,p} \le \max_{|q|\le p} \|\partial^q \chi\|_\infty$ and so $\varphi_m(x) \longrightarrow M_p(x)^{-1}$ uniformly in the seminorm $\|\cdot\|_{\alpha,p}$. Thus

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{M_p(x)}\,dx \le C_p \max_{|q|\le p} \|\partial^q \chi\|_\infty < +\infty.$$

■

We now know a bit about the structure of $\mathscr{S}_\alpha(\mathbb{R}^n)$ in terms of its operations and functionals on the space. To state and prove a Krein theorem for this space we will need to know something about the Fourier transform of functions in $\mathscr{S}_\alpha(\mathbb{R}^n)$. More specifically, we will need to know the asymptotic behaviour of such functions. Define the closed tube

$$T_\alpha = \Big\{ z \in \mathbb{C}^n : \|\mathrm{Im}(z)\| \le \alpha \Big\}$$

and consider the multiplicative $*$-algebra $\mathscr{S}(T_\alpha)$ of holomorphic functions h on the interior of T_α whose derivatives all extend continuously to T_α, and is bounded by the seminorms

$$\|h\|^{\alpha,p} = \max_{|q|\le p} \sup_{z\in T_\alpha} N_p(z)|\partial^q h(z)|\,, \quad N_p(z) = (1+\|z\|^2)^p\,.$$

Other notations for this space are $\mathscr{S}^\alpha(\mathbb{R}^n)$ in [9] and $\mathscr{Z}_\alpha(\mathbb{C}^n)$ in [10]. The purpose of this space is that it characterizes the Fourier transform of functions in $\mathscr{S}_\alpha(\mathbb{R}^n)$.

Lemma 5.9. *The Fourier transform $\mathscr{F} : C_c^\infty(\mathbb{R}^n) \to \mathrm{PW}(\mathbb{C}^n)$ extends to a $*$-isomorphism $\mathscr{S}_\alpha(\mathbb{R}^n) \to \mathscr{S}(T_\alpha)$.*

Proof. If $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ then $\varphi(x)\mathrm{e}^{-i\langle x,z\rangle}$ is holomorphic in $z \in \mathbb{C}^n$ for all $x \in \mathbb{R}^n$. Moreover,

$$|\varphi(x)\mathrm{e}^{-i\langle x,z\rangle}| \le |\varphi(x)|\mathrm{e}^{\|\mathrm{Im}(z)\|\|x\|} \le |\varphi(x)|\mathrm{e}^{\alpha\|x\|}$$

and since the RHS is integrable then for all triangles $\Delta \subset \mathbb{C}$,

$$\int_\Delta \widehat{\varphi}(z)\,dz_j = \int_{\mathbb{R}} \varphi(x) \int_\Delta e^{-i\langle x,z\rangle}\,dz_j dx = 0$$

for all j. By Morera's theorem, $\widehat{\varphi}$ is holomorphic in all z_j, hence in z. Moreover, for large $k \in \mathbb{N}$,

$$\sup_{z\in T_\alpha} |\widehat{\varphi}(z)| \le \int_{\mathbb{R}^n} |\varphi(x)| e^{\langle \mathrm{Im}(z),x\rangle}\,dx \le \int_{\mathbb{R}^n} |\varphi(x)| e^{\alpha\|x\|}\,dx \le \|\varphi\|_{\alpha,k} \int_{\mathbb{R}^n} \frac{dx}{(1+\|x\|^2)^k}.$$

Using that linear differential operators with constant coefficients transform to multiplication operators by a polynomial, then

$$\|\widehat{\varphi}\|^{\alpha,p} \le \|(1+\Delta)^p \varphi\|_{\alpha,k} \int_{\mathbb{R}^n} \frac{dx}{(1+\|x\|^2)^k} \le \|\varphi\|_{\alpha,2p+k} \int_{\mathbb{R}^n} \frac{dx}{(1+\|x\|^2)^k} < +\infty,$$

where Δ this time denotes the Laplace operator. This means that we have a $*$-homomorphism $\mathscr{F} : \mathscr{S}_\alpha(\mathbb{R}^n) \to \mathscr{S}(T_\alpha)$, and to show that it is an isomorphism it suffices to show that the inverse $\mathscr{F}^{-1}$ is well-defined and continuous. First observe that integration around the boundary of a rectangle in $\mathbb{C}^n$ is zero by Cauchy's theorem, so in particular for $h \in \mathscr{S}(T_\alpha)$ and $\theta \in [0,\alpha]$,

$$0 = \int_{-R}^{R} h(x)\,dx_j + \int_0^\theta h(Re_j + iy)\,dy_j - \int_{-R}^{R} h(x+i\theta e_j)\,dx_j - \int_0^\theta h(-Re_j+iy)\,dy_j$$

for all basis vectors $e_j \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$. The restriction of h to $\mathbb{R}^n$ is a Schwartz function, so $\varphi = \mathscr{F}^{-1}h$ is a well-defined Schwartz function and if we let $R \to +\infty$ then

$$\begin{aligned}\int_{\mathbb{R}} h(y) e^{ix_j y_j}\,dy_j &= \int_{\mathbb{R}^n} h(y+i\theta e_j) e^{ix_j(y_j+i\theta)}\,dy_j \\ &= e^{-x_j\theta} \int_{\mathbb{R}} h(y+i\theta e_j) e^{ix_j y_j}\,dy_j.\end{aligned}$$

This means that for every $\theta \in \overline{B(0,\alpha)}$,

$$\begin{aligned}\varphi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} h(y) e^{-i\langle x,y\rangle}\,dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(y) e^{-ix_1,y_1}\,dy_1 \ldots e^{-ix_n,y_n}\,dy_n \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} h(y+i\theta) e^{-ix_1(y_1+i\theta_1)}\,dy_1 \ldots e^{-ix_n(y_n+i\theta_n)}\,dy_n \\ &= \frac{e^{\langle x,\theta\rangle}}{(2\pi)^n} \int_{\mathbb{R}^n} h(y+i\theta) e^{-i\langle x,y\rangle}\,dy\end{aligned}$$

and for any $k \ge n$,

$$\begin{aligned}|e^{-\langle x,\theta\rangle}\varphi(x)| &\le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |h(y+i\theta)|\,dy \\ &\le \frac{1}{(2\pi)^n} \|h\|^{\alpha,k} \int_{\mathbb{R}^n} \frac{dx}{N_k(x+i\theta)} \\ &\le \frac{1}{(2\pi)^n} \|h\|^{\alpha,k} \int_{\mathbb{R}^n} \frac{dx}{N_k(x)} < +\infty.\end{aligned}$$

Taking $\|\theta\| = \alpha$ such that $\langle x,\theta\rangle = \alpha\|x\|$ and supremum over $x \in \mathbb{R}^n$ we obtain $\|\varphi\|_{\alpha,0} \le C_k \|h\|^{\alpha,k}$, where

$$C_k = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{dx}{N_k(x)} < +\infty.$$

Applying derivatives of order less than or equal to $p \in \mathbb{N}$ we get that $\|\varphi\|_{\alpha,p} \le C_{k+p} \|h\|^{\alpha,k+p}$, so $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ and the inverse Fourier transform $\mathscr{F}^{-1} : \mathscr{S}(T_\alpha) \to \mathscr{S}_\alpha(\mathbb{R}^n)$ is continuous.

■

Corollary 5.10. *Let $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$. Then there is a constant $C_{\alpha,\varphi}$ such that for all $z \in T_\alpha$,*

$$|\widehat{\varphi}(z)| \leq \frac{C_{\alpha,\varphi}}{1+\|\mathrm{Re}(z)\|^2}.$$

The uniqueness of the Krein measure from the Gelfand-Shilov space will follow from the fact that the functions $\widehat{\varphi}$ vanish at infinity on T_α.

Theorem 5.11. *(Gelfand-Vilenkin-Krein, uniqueness of measure) If $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ is evenly positive-definite then the Krein measure μ_ξ is uniquely defined and is supported on*

$$(\mathbb{R} \cup i\mathbb{R})^n \cap T_\alpha = \left\{ z \in \mathbb{C}^n : z_j \in \mathbb{R}^n \ \ or \ \ z_j \in i\overline{B(0,\alpha)} \right\}$$

The classification of evenly positive-definite distributions was in 1979 extended by N. Bopp in [6, p. 15-50] to finite subgroups $W < \mathrm{O}(n)$. To state the theorem, define the set

$$\Omega_W = \left\{ z \in \mathbb{C}^n : \overline{z} = w.z \text{ for some } w \in W \right\}.$$

Note that if we take $W = \mathrm{O}(1)^n$, then $\Omega_W = (\mathbb{R} \cup i\mathbb{R})^n$ and W-positive-definiteness corresponds to being evenly positive-definite.

Theorem 5.12. *(Bopp-Gelfand-Vilenkin-Krein) Let $W < \mathrm{O}(n)$ be finite. Then a distribution $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ is W-positive-definite if and only if it is the Fourier transform of a unique positive W-invariant Radon measure μ_ξ on $\Omega_W \cap T_\alpha$.*

We will dedicate the next section to a proof of this theorem. A special case of the theorem is when $\alpha = 0$ and ξ is a tempered distribution. Then $\Omega_W \cap T_0 = \mathbb{R}^n$.

Corollary 5.13. *A distribution $\xi \in \mathscr{S}(\mathbb{R}^n)^*$ is W-positive-definite if and only if it is the Fourier transform of a unique positive W-invariant Radon measure μ_ξ on $\mathbb{R}^n$.*

6 A Proof of Krein's Theorem for Finite Subgroups of $\mathrm{O}(n)$

In this part of the thesis we will present Bopp's proof theorem 5.12 of Bopp-Gelfand-Vilenkin-Krein for the $*$-algebra $\mathscr{S}_\alpha(\mathbb{R}^n)^W$. The main idea of the proof is to

1. identify the spectrum of characters

$$X_{\alpha,W} = \mathrm{Spec}\, \mathscr{S}_\alpha(\mathbb{R}^n)^W = \left\{ \omega \in (\mathscr{S}_\alpha(\mathbb{R}^n)^W)^* : \omega(\varphi * \psi^*) = \omega(\varphi)\overline{\omega(\psi)} \right\}$$

   with the quotient space $W \backslash (\Omega_W \cap T_\alpha)$, and
2. using the Plancherel-Godement theorem, found in appendix B, construct a positive Radon measure ν_ξ from ξ on $X_{\alpha,W}$.

We can then lift ν_ξ to a positive W-invariant Radon measure μ_ξ on $\Omega_W \cap T_\alpha$.

6.1 Identifying the Spectrum

To every $z \in \mathbb{C}^n$ we can associate a unique character $\chi_z(x) = \mathrm{e}^{-i\langle x,z\rangle}$. It is locally integrable and so there is a well-defined map

$$\begin{aligned} \chi : \mathbb{C}^n &\longrightarrow C_c^\infty(\mathbb{R}^n)^* \\ z &\longmapsto \chi_z \end{aligned}.$$

The objective in identifying the spectrum here will be, using the above map, to construct a homeomorphism $W\backslash(\Omega_W \cap T_\alpha) \longrightarrow X_{\alpha,W}$. The forementioned characters extend to distributions in $\mathscr{S}_\alpha(\mathbb{R}^n)^*$ if they satisfy the condition in lemma 5.8. For $z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$, $|\chi_z(x)| = \mathrm{e}^{\langle \mathrm{Im}(z), x\rangle} \leq \mathrm{e}^{\|\mathrm{Im}(z)\|\|x\|}$ with equality when x and $\mathrm{Im}(z)$ are parallell, so χ_z defines a distribution on $\mathscr{S}_\alpha(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \frac{\mathrm{e}^{(\|\mathrm{Im}(z)\|-\alpha)\|x\|}}{(1+\|x\|^2)^p} dx < +\infty,$$

i.e. $z \in T_\alpha$. This means that we have a map

$$\begin{aligned} \chi : T_\alpha &\longrightarrow \mathscr{S}_\alpha(\mathbb{R}^n)^* \\ z &\longmapsto \chi_z \quad . \end{aligned}$$

A W-positive-definite distribution $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ defines a positive-definite distribution on the test function space $\mathscr{S}_\alpha(\mathbb{R}^n)^W$ and we will view ξ as such a distribution from now on. To put the characters χ_z, $z \in T_\alpha$ into this context we average them over the action of W, yielding *Bessel functions*

$$q_z(x) = \frac{1}{|W|} \sum_{w \in W} \chi_z(w.x).$$

From this we obtain a new map

$$\begin{aligned} q : W\backslash T_\alpha &\longrightarrow (\mathscr{S}_\alpha(\mathbb{R}^n)^W)^* \\ z &\longmapsto q_z \quad . \end{aligned}$$

This map will be our candidate for the sought after homeomorphism $W\backslash(\Omega_W \cap T_\alpha) \longrightarrow X_{\alpha,W}$. First, let us show the necessity of the domain.

Lemma 6.1. *Let $z \in T_\alpha$. Then the function q_z determines a character in $X_{\alpha,W}$ if and only if $z \in \Omega_W$.*

If $z \in \Omega_W$ then $q_z(\varphi * \psi^*) = \widehat{\varphi}(z)\overline{\widehat{\psi}(\overline{z})} = \widehat{\varphi}(z)\overline{\widehat{\psi}(z)} = q_z(\varphi)\overline{q_z(\psi)}$ for all $\varphi, \psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$, so it remains to show the "only if"-part of the statement. To do this we turn to invariant theory.

Theorem 6.2. *Every algebra homomorphism $\mathbb{C}[X_1,...,X_n]^W \longrightarrow \mathbb{C}$ is an evaluation at some unique $z \in \mathbb{C}^n$, up to the action of W.*

Remark. It is here that the finiteness of W comes into play, since the above theorem does not hold in general for subgroups $W < \mathrm{O}(n)$. We refer to appendix A for a proof of theorem 6.2 and discussion on this.

Proof. (Proof of lemma 6.1.) If q_z is a character in $X_{\alpha,W}$ then it is in particular a positive-definite continuous function on $\mathbb{R}^n$, so $q_z^*(x) = \overline{q_z(-x)} = q_z(x)$ by corollary 3.3. But at the same time, $q_z^*(x) = q_{\overline{z}}(x)$, so

$$P(iz) = P(\partial)q_z(0) = P(i\overline{z})$$

for all $P \in \mathbb{C}[X_1,...,X_n]^W$. Thus by theorem 6.2 we must have that $\overline{z} = w.z$ for some $w \in W$, i.e. $z \in \Omega_W$. ■

Now we have a well-defined map

$$\begin{aligned} q : W\backslash(\Omega_W \cap T_\alpha) &\longrightarrow X_{\alpha,W} \\ W.z &\longmapsto q_z \end{aligned}$$

and it remains to prove that this is a homeomorphism. By the same reasoning as in the proof of lemma 6.1 $q_z = q_{z'}$ implies that $z' \in W.z$, so q is injective. Let us prove the surjectivity. To do this we first construct an algebra homomorphism $\mathbb{C}[X_1,...,X_n]^W \to \mathbb{C}$.

Lemma 6.3. *Let $\omega \in X_{\alpha,W}$ and $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ such that $\omega(\varphi) \neq 0$. Then the map $\lambda_\omega : \mathbb{C}[X_1,...,X_n]^W \to \mathbb{C}$ defined by*

$$\lambda_\omega(P) = \frac{\omega(P(\partial)\varphi)}{\omega(\varphi)}$$

is a well-defined algebra homomorphism.

Proof. If $\varphi, \psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ then

$$\omega(P(\partial)\varphi)\omega(\psi) = \omega(P(\partial)(\varphi * \psi)) = \omega(\varphi)\omega(P(\partial)\psi),$$

so if φ, ψ both are functions such that $\omega(\varphi), \omega(\psi) \neq 0$ then

$$\frac{\omega(P(\partial)\varphi)}{\omega(\varphi)} = \frac{\omega(P(\partial)\psi)}{\omega(\psi)}.$$

Thus λ_ω is well-defined. It moreover follows from the equation above that if $\omega(P(\partial)\varphi) = 0$ then $\omega(P(\partial)\psi) = 0$ for all ψ such that $\omega(\psi) \neq 0$. The kernel of a non-zero functional on a vector space is a proper vector subspace, so it's complement

$$\left\{\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W : \omega(\varphi) \neq 0\right\}$$

is dense in $\mathscr{S}_\alpha(\mathbb{R}^n)^W$. It therefore follows that $\omega(P(\partial)\psi) = 0$ for any $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ whenever $\omega(P(\partial)\varphi) = 0$. Now, if $P, Q \in \mathbb{C}[X_1,...,X_n]^W$ and $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ such that $\omega(\varphi) \neq 0$, then

$$\lambda_\omega(PQ) = \frac{\omega(P(\partial)Q(\partial)\varphi)}{\omega(Q(\partial)\varphi)} \frac{\omega(Q(\partial)\varphi)}{\omega(\varphi)} = \lambda_\omega(P)\lambda_\omega(Q)$$

whenever $\omega(Q(\partial)\varphi) \neq 0$. Otherwise, if $\omega(Q(\partial)\varphi) = 0$, then $\omega(Q(\partial)\psi) = 0$ for all $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$. In particular we have that $\omega(Q(\partial)P(\partial)\varphi) = 0$ and so

$$\lambda_\omega(PQ) = \frac{\omega(Q(\partial)P(\partial)\varphi)}{\omega(\varphi)} = 0 = \lambda_\omega(P)\frac{\omega(Q(\partial)\varphi)}{\omega(\varphi)} = \lambda_\omega(P)\lambda_\omega(Q).$$

■

The definition of $\lambda_\omega(P)$ may equivalently be written as $P(\partial)\omega = \lambda_\omega(P)\omega$, so that ω becomes an eigendistribution of $P(\partial)$. By theorem 6.2 we know that there is a $z \in \mathbb{C}^n$ such that λ_ω is evaluation at z and for any $z \in \mathbb{C}^n$ we have that q_z is an eigendistribution of $P(\partial)$ with eigenvalue $P(iz)$, so to prove surjectivity of the map $q : W\backslash(\Omega_W \cap T_\alpha) \longrightarrow X_{\alpha,W}$ we need to prove the following lemma:

Lemma 6.4. *If $\omega \in X_{\alpha,W}$ is an eigendistribution of $P(\partial)$ with eigenvalue $P(iz)$ for some $z \in \mathbb{C}^n$ and every $P \in \mathbb{C}[X_1,..,X_n]^W$, then $z \in \Omega_W \cap T_\alpha$ and $\omega = q_z$ in $X_{\alpha,W}$.*

Proof. Let $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ and consider the smooth function $\omega * \check{\varphi}$ on $\mathbb{R}^n$. Then we have

$$\begin{aligned}(\omega * \check{\varphi})(w.x) &= \omega(\varphi(\cdot - w.x)) \\ &= \omega(\varphi(w.(\cdot - x))) \\ &= \omega(\varphi(\cdot - x)) \\ &= (\omega * \check{\varphi})(x)\end{aligned}$$

for all $w \in W$, so $\omega * \check{\varphi} \in C^\infty(\mathbb{R}^n)^W$. If $P \in \mathbb{C}[X_1,...,X_n]^W$ then by corollary 5.7

$$P(\partial)(\omega * \check{\varphi}) = P(\partial)\omega * \check{\varphi} = P(iz)(\omega * \check{\varphi}),$$

and for every multiindex $q \in \mathbb{N}_0^n$ we write P_q for the W-invariant polynomial associated to the differential operator

$$(\partial^q)^W f(x) = \frac{1}{|W|} \sum_{w \in W} \partial^q f(w.x).$$

It satisfies $|P_q(z)| \le C_q \|z\|^q$ for some $C_q \ge 0$ and for any $x \in \mathbb{R}^n$ we have

$$\partial^q(\omega * \check{\varphi}) = (\partial^q)^W(\omega * \check{\varphi}) = P_q(iz)(\omega * \check{\varphi}).$$

We want to show that ω defines an analytic function on $\mathbb{R}^n$, and then the uniqueness will follow. Fix $x \in \mathbb{R}^n$. The the remainder terms of the Maclaurin polynomials of $\omega * \check{\varphi}$ are of the form

$$R_q(x) = \frac{\partial^q(\omega * \check{\varphi})(\theta_x)}{q!} x^q = (\omega * \check{\varphi})(\theta_x) \frac{P_q(iz)}{q!} x^q ,$$

for some $\theta_x \in B(0, \|x\|)$. As in the proof of lemma 5.6, equation 5.1, $|(\omega * \check{\varphi})(x)| \le C_\omega e^{\alpha\|x\|} \|\varphi\|_{\alpha,0}$ for some $C_\omega \ge 0$, so

$$|R_q(x)| \le C_\omega e^{\alpha\|x\|} \|\varphi\|_{\alpha,0} C_q \frac{(\|z\|\,\|x\|)^{|q|}}{q!}.$$

Applying the root or ratio test for these errors, one sees that the Maclaurin series of $\omega * \check{\varphi}$ converges to the function on the open ball $B(0, \|x\|)$. As $x \in \mathbb{R}^n$ was arbitrary, we get that $\omega * \check{\varphi}$ is real analytic on $\mathbb{R}^n$ and we write

$$(\omega * \check{\varphi})(x) = \omega(\varphi) \sum_{q \in \mathbb{N}_0^n} \frac{P_q(iz)}{q!} x^q ,$$

using that $(\omega * \check{\varphi})(0) = \omega(\varphi)$. Taking a radial approximate identity ρ_ε in place of φ then $\omega(\psi)\omega(\rho_\varepsilon) = \omega(\psi * \rho_\varepsilon) \longrightarrow \omega(\psi)$ for any ψ, so $\lim_{\varepsilon\to 0} \omega(\rho_\varepsilon) = 1$ as ω is non-trivial. Thus if we define the function

$$f_\omega(x) = \lim_{\varepsilon \to 0} (\omega * \check{\rho}_\varepsilon)(x) = \sum_{q \in \mathbb{N}_0^n} \frac{P_q(iz)}{q!} x^q ,$$

then f_ω is real-analytic and ω acts on $\mathscr{S}_\alpha(\mathbb{R}^n)^W$ by integration against f_ω. Note that we only used that $P(\partial)\omega = P(iz)\omega$ for all $P \in \mathbb{C}[X_1, ..., X_n]^W$, so everything above holds in particular for the distribution $q_z \in X_{\alpha,W}$. Thus $f_{q_z} = f_\omega$ and as $\omega \in X_{\alpha,W}$ and $q_z = \omega$ as distributions in $X_{\alpha,W}$. That $q_z \in X_{\alpha,W}$ is by lemma 5.10 and lemma 6.1 equivalent to $z \in \Omega_W \cap T_\alpha$. ∎

It remains to show that the maps q and q^{-1} are continuous.

Theorem 6.5. *The map*

$$\begin{aligned} q : W\backslash(\Omega_W \cap T_\alpha) &\longrightarrow X_{\alpha,W} \\ W.z &\longmapsto q_z \end{aligned}$$

is a homeomorphism.

Proof. To see this, consider the one-point compactification of both spaces and the induced map

$$\tilde{q} : W\backslash(\Omega_W \cap T_\alpha) \cup \{\infty\} \longrightarrow X_{\alpha,W} \cup \{0\}$$

by sending the added point at infinity to the zero character. Then as $X_{\alpha,W}$ is Hausdorff it suffices to show that $\tilde{q}$ is continuous and since $W\backslash(\Omega_W \cap T_\alpha)$ is metrizable, take for example the metric $d(W.z, W.z') = \inf_{w\in W} \|w.z - z'\|$, then it suffices to show that for every sequence z_n in $\Omega_W \cap T_\alpha$ with limit z in $(\Omega_W \cap T_\alpha) \cup \{\infty\}$ we have $q_{z_n} \to q_z$ in $X_{\alpha,W}$. Note that $q_z(\varphi) = \widehat{\varphi}(z)$, so by continuity of the Fourier transform it is clear that this holds for $z \ne \infty$. If $z = \infty$ then $z_n \to \infty$ in T_α and by corollary 5.10,

$$|\widehat{\varphi}(z_n)| \le \frac{C_{\alpha,\varphi}}{1 + \|\mathrm{Re}(z_n)\|^2} \xrightarrow[n]{} 0$$

for every $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$, so $q_{z_n} \to 0$.

■

6.2 Constructing a Measure

Let $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ be a W-positive-definite distribution. Then ξ by definition defines a continuous $*$-positive functional on the $*$-algebra $\mathscr{S}_\alpha(\mathbb{R}^n)^W$ of W-invariant functions in $\mathscr{S}_\alpha(\mathbb{R}^n)$. The following result due to R. Godement provides a context for when a $*$-positive functional can be expressed in terms of a positive Radon measure on spectra.

Theorem 6.6. *(Plancherel-Godement) Let A be a topological commutative $*$-algebra with an approximate identity, and let $\alpha \in A^*$ be a $*$-positive functional satisfying*

$$\alpha(xyy^*x^*) \le k_y \alpha(xx^*), \quad k_y \ge 0,$$

for every $x, y \in A$. Moreover, let $\mathfrak{p}_A$ be the dense ideal in A generated by all elements of the form yy^*, $y \in A$. Then there is a unique positive Radon measure ν_α on a locally compact subspace $\Omega_\alpha \subset \operatorname{Spec} A$ such that for every $x \in \mathfrak{p}_A$ we have

$$\alpha(x) = \int_{\Omega_\alpha} \chi(x)\, d\nu_\alpha(\chi).$$

Let us prove that the functional $\xi \in (\mathscr{S}_\alpha(\mathbb{R}^n)^W)^*$ satisfies the inequality stated in the Plancherel-Godement theorem. To do it, we will first need to know something about bounds of continuous W-positive-definite functions.

Lemma 6.7. *Let $f \in C(\mathbb{R}^n)$ be W-positive-definite such that $f(0) \ge 0$. If there are constants $C \ge 0$ and $a \ge 0$ such that*

$$|f(x)| \le C e^{a\|x\|}$$

then

$$|f(x)| \le f(0) e^{a\|x\|}.$$

Proof. Define the constant $C_f = \sup_{x \in \mathbb{R}^n} |f(x)| e^{-a\|x\|}$, which by assumption is less or equal to C and hence finite. Clearly, $|f(x)| \le C_f e^{a\|x\|}$ so it suffices to show that $C_f \le f(0)$. A W-positive-definite function f defines a positive hermitian form h_f as described above, so by the Cauchy-Schwarz inequality,

$$\left| \int_{\mathbb{R}^n} (\varphi * \psi^*)(x) f(x)\, dx \right|^2 \le \int_{\mathbb{R}^n} (\varphi * \varphi^*)(x) f(x)\, dx \int_{\mathbb{R}^n} (\psi * \psi^*)(x) f(x)\, dx.$$

Taking $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)^W$ to be an approximate identity in place of ψ we get that

$$\left| \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx \right|^2 \le f(0) \int_{\mathbb{R}^n} (\varphi * \varphi^*)(x) f(x)\, dx.$$

Now take $x_0 \in \mathbb{R}^n$ and define

$$\rho_{\varepsilon, x_0}(x) = \frac{1}{|W|} \sum_{w \in W} \rho_\varepsilon(x - w.x_0),$$

which tends to a Dirac comb on $W.x_0$ as $\varepsilon \to 0$. In particular, as f is W-invariant then

$$\int_{\mathbb{R}^n} \rho_{\varepsilon, x_0}(x) f(x)\, dx \longrightarrow f(x_0).$$

The support of ρ_{ε, x_0} is contained in the ball centered at 0 of radius $2(\|x_0\| + \varepsilon)$ as

$$\operatorname{supp}(\rho_{\varepsilon,x_0} * \rho^*_{\varepsilon,x_0}) \subset \operatorname{supp}(\rho_{\varepsilon,x_0}) + \operatorname{supp}(\rho^*_{\varepsilon,x_0}) \subset B_{2\varepsilon}(W.x_0) + B_{2\varepsilon}(0) \subset B_{2(\|x_0\|+\varepsilon)}(0),$$

and by the assumption of the lemma we must have that

$$\int_{\mathbb{R}^n} (\rho_{\varepsilon,x_0} * \rho^*_{\varepsilon,x_0})(x) f(x)\, dx \le C_f e^{2\alpha(\|x_0\|+\varepsilon)}.$$

By the Cauchy-Schwarz inequality,

$$\left|\int_{\mathbb{R}^n} \rho_{\varepsilon,x_0}(x) f(x)\, dx\right|^2 \le f(0)\int_{\mathbb{R}^n} (\rho_{\varepsilon,x_0} * \rho^*_{\varepsilon,x_0})(x) f(x)\, dx \le f(0) C_f e^{2\alpha(\|x_0\|+\varepsilon)}$$

and so as $\varepsilon \to 0$ then $|f(x_0)|^2 \le f(0) C_f e^{2\alpha\|x_0\|}$. Moreover, $|f(x_0)| \le C_f e^{\alpha\|x_0\|}$ and this bound is minimal by the definition of supremum, so

$$C_f \le \sqrt{f(0) C_f}.$$

If C_f is nonzero then dividing by $\sqrt{C_f}$ and squaring both sides yields $C_f \le f(0)$. ■

Lemma 6.8. *For any $\varphi, \psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ we have that*

$$\xi(\varphi * \psi * \psi^* * \varphi^*) \le \xi(\psi * \psi^*)\Big(\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi(x)|\, dx\Big)^2.$$

Proof. Let $\psi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$. From lemma 5.6 and corollary 5.7 we can consider the smooth function $\xi_\psi = \xi * \overline{\psi} * \check{\psi}$ on $\mathbb{R}^n$. It defines a continuous functional on $\mathscr{S}_\alpha(\mathbb{R}^n)$ and it is moreover W-positive-definite as

$$\int_{\mathbb{R}^n} (\xi * \overline{\psi} * \check{\psi})(x)(\varphi * \varphi^*)(x)\, dx = \xi(\psi * \psi^* * \varphi * \varphi^*) = \xi((\varphi * \psi) * (\varphi * \psi)^*) \ge 0.$$

Fix $x \in \mathbb{R}^n$. Then there is a constant $C_\xi \ge 0$ such that

$$|\xi_\psi(x)| = |\xi(\tau_x(\psi * \psi^*))| \le C_\xi \left\|\psi * \psi^*\right\|_{\alpha,0} e^{\alpha\|x\|} \le C_\xi \left\|\psi\right\|^2_{\alpha,0} e^{\alpha\|x\|}$$

by equation 5.1 and proposition 5.5. By lemma 6.7, $|\xi_\psi(x)| \le \xi_\psi(0) e^{\alpha\|x\|} = \xi(\psi * \psi^*) e^{\alpha\|x\|}$ and by lemma 5.6 we have

$$\begin{aligned}\xi(\varphi * \psi * \psi^* * \varphi^*) &= \int_{\mathbb{R}^n} \xi_\psi(x)(\varphi * \varphi^*)(x)\, dx \\ &\le \xi(\psi * \psi^*)\int_{\mathbb{R}^n} e^{\alpha\|x\|} |(\varphi * \varphi^*)(x)|\, dx.\end{aligned}$$

To finish the proof, note that

$$\begin{aligned}\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi * \varphi^*(x)|\, dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi(x+y)||\varphi(y)|\, dy dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\alpha\|x-y\|} |\varphi(x)||\varphi(y)|\, dy dx \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi(x)| e^{\alpha\|y\|} |\varphi(y)|\, dy dx \\ &= \Big(\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi(x)|\, dx\Big)^2.\end{aligned}$$

■

We have now shown that ξ satisfies the criteria of the Plancherel-Godement theorem by taking $k_\varphi = \left(\int_{\mathbb{R}^n} e^{\alpha\|x\|} |\varphi(x)|\, dx\right)^2$. The ideal $\mathfrak{p}_{\alpha,W} \subset \mathscr{S}_\alpha(\mathbb{R}^n)^W$ generated by functions $\varphi * \varphi^*$, $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$

contains every

$$\varphi * \psi^* = \sum_{i=0}^{3} i^k (\varphi + i^k \psi) * (\varphi + i^k \psi)^*$$

and by the presence of an approximate identity $\rho_\varepsilon \in \mathscr{S}_\alpha(\mathbb{R}^n)^W$ we see that $\varphi * \rho_\varepsilon^* \to \varphi$ in $\mathscr{S}_\alpha(\mathbb{R}^n)^W$, so $\mathfrak{p}_{\alpha,W}$ is dense in $\mathscr{S}_\alpha(\mathbb{R}^n)^W$. This argument holds for a general algebra A as in the theorem. We now have a positive Radon measure ν_ξ supported on the locally compact space $X_{\alpha,W}$ and the integral representation

$$\xi(\varphi) = \int_{X_{\alpha,W}} \omega(\varphi)\, d\nu_\xi(\omega)$$

for all $\varphi \in \mathfrak{p}_{\alpha,W}$. The homeomorphism $q : W\backslash(\Omega_W \cap T_\alpha) \to X_{\alpha,W}$ can be used to push the measure ν_ξ forward to a positive Radon measure $(q^{-1})_* \nu_\xi$ on $W\backslash(\Omega_W \cap T_\alpha)$, and it lifts to a W-invariant positive Radon measure μ_ξ on $\Omega_W \cap T_\alpha \subset \mathbb{C}^n$. We now have that

$$\xi(\varphi) = \int_{X_{\alpha,W}} \omega(\varphi)\, d\nu_\xi(\omega) = \int_{W\backslash(\Omega_W \cap T_\alpha)} q_z(\varphi)\, d(q^{-1})_* \nu_\xi(W.z) = \int_{\Omega_W \cap T_\alpha} \widehat{\varphi}(z)\, d\mu_\xi(z)$$

for every $\varphi \in \mathfrak{p}_{\alpha,W}$. We can moreover preserve this equality for all elements of the ideal $\mathfrak{p}_\alpha \subset \mathscr{S}_\alpha(\mathbb{R}^n)$, generated by all $\varphi * \varphi^*$, $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ by acting on W-invariant parts. To extend this integral representation to all of $\mathscr{S}_\alpha(\mathbb{R}^n)$, we need to prove that the measure μ_ξ defines a functional on all of $\mathscr{S}(T_\alpha)$.

Lemma 6.9. *Let $\xi \in \mathscr{S}_\alpha(\mathbb{R}^n)^*$ be W-positive-definite. Then the measure μ_ξ is tempered in the sense that there is a $p \in \mathbb{N}$ such that*

$$\int_{\Omega_W \cap T_\alpha} \frac{d\mu_\xi(z)}{(1 + \|z\|^2)^p} < +\infty.$$

Proof. We first observe that the continuity of ξ the Fourier transform implies that there is a $k \in \mathbb{N}$ and a constant $C_k > 0$ such that for every $\varphi \in \mathscr{S}_\alpha(\mathbb{R}^n)$ with Fourier transform $h \in \mathscr{S}(T_\alpha)$,

$$\int_{\Omega_W \cap T_\alpha} |h|^2\, d\mu_\xi = \xi(\varphi * \varphi^*) \le C_k \left\| |h|^2 \right\|^{\alpha,k}.$$

We will construct a bounded sequence $h_m \in \mathscr{S}(T_\alpha)$ such that

$$|h_m(z)|^2 \longrightarrow \frac{1}{(c_\alpha + P(z))^p}$$

uniformly on compacta for some $p \in \mathbb{N}$, where $c_\alpha > \max(1, \alpha^2)$ and $P(X) = X_1^2 + \ldots + X_n^2 \in \mathbb{C}[X_1, \ldots, X_n]^W$ on $\mathbb{C}^n$. Note that $P(x+iy) = P(x-iy) = \|x\|^2 - \|y\|^2 \ge -\alpha^2$ for all $x+iy \in \Omega_W$, so $c_\alpha + P(z) > 0$ for all $z \in \Omega_W$. Thus, for such a $p \in \mathbb{N}$,

$$0 \le \int_{\Omega_W \cap T_\alpha} \frac{d\mu_\xi(z)}{(c_\alpha + \|z\|^2)^p} \le \int_{\Omega_W \cap T_\alpha} \frac{d\mu_\xi(z)}{(c_\alpha + P(z))^p} = \lim_m \int_{\Omega_W \cap T_\alpha} |h_m|^2 d\mu_\xi < +\infty.$$

This means that, up to a constant, μ_ξ is tempered.

For the construction, take a function $h \in \mathscr{S}(T_\alpha)$ such that $h(0) = 1$ and let

$$h_m(z) = \frac{h(z/m)}{(c_\alpha + P(z))^k}.$$

It is clear that $h_m(z) \to (c_\alpha + P(z))^{-2k}$ uniformly on compacta. Using the product rule one has

$$\partial^q h_m(z) = \sum_{r \le q} (-2)^{|r|} \frac{(p + |r| - 1)!}{(p-1)!} \frac{z^r}{(c_\alpha + P(z))^{2k+|r|}} m^{|r|-|q|} \partial^{q-r} h(z/m),$$

and so for every multiindex q with $|q| \le 2k$,

$$(1+\|z\|^2)^{2k}|\partial^q h_m(z)| \le \max_{0\le r\le|q|}\sup_{z'\in T_\alpha}|\partial^q h(z')|\sum_{r\le q}(-2)^{|r|}\frac{(p+|r|-1)!}{(p-1)!}\frac{(1+\|z\|^2)^{2k}z^r}{(c_\alpha+P(z))^{2k+|r|}}.$$

Note that

$$\frac{(1+\|z\|^2)^{2k}z^r}{(c_\alpha+P(z))^{2k+|r|}} \longrightarrow \begin{cases}1 \text{ if } r=0\\ 0 \text{ if } 0<r\le q\end{cases}$$

as $z\to\infty$ in $\Omega_W\cap T_\alpha$, so $(1+\|z\|^2)^{2k}|\partial^q h_m(z)|$ stays bounded on $\Omega_W\cap T_\alpha$. Thus there is a constant $M_{k,q}\ge 0$ such that $(1+\|z\|^2)^k|\partial^q h_m(z)| \le M_{k,q}$, and taking $M_k = \max_{|q|\le k} M_{k,q}$ we get that

$$\int_{\Omega_W\cap T_\alpha}\frac{d\mu_\xi(z)}{(c_\alpha+P(z))^{2k}} = \lim_m \int_{\Omega_W\cap T_\alpha}|h_m|^2 d\mu_\xi \le C_k M_k < +\infty.$$

■

We can now show that the integral representation of ξ in terms of μ_ξ extends to all of $\mathscr{S}_\alpha(\mathbb{R}^n)$. If $h\in\mathscr{S}(T_\alpha)$ and we take a $p\in\mathbb{N}$ as in lemma 6.9 then

$$|\mu_\xi(h)| \le \int_{\Omega_W\cap T_\alpha}|h(z)|d\mu_\xi(z) \le \|h\|^{\alpha,p}\int_{\Omega_W\cap T_\alpha}\frac{d\mu_\xi(z)}{(1+\|z\|^2)^p} < +\infty,$$

meaning that $\mu_\xi\in\mathscr{S}(T_\alpha)^*$. Letting $\rho_\varepsilon\in\mathscr{S}_\alpha(\mathbb{R}^n)$ be an approximate identity as before, then $\varphi*\rho_\varepsilon^*\to\varphi$ in $\mathscr{S}_\alpha(\mathbb{R}^n)$ and $\widehat{\varphi}\overline{\widehat{\rho}_\varepsilon} = \widehat{\varphi*\rho_\varepsilon^*}\to\widehat{\varphi}$ in $\mathscr{S}(T_\alpha)$ by continuity of the Fourier transform. Finally we have

$$\xi(\varphi) = \lim_\varepsilon \xi(\varphi*\rho_\varepsilon^*) = \lim_\varepsilon\int_{\Omega_W\cap T_\alpha}\widehat{\varphi}(z)\overline{\widehat{\rho}_\varepsilon(z)}\,d\mu_\xi(z) = \int_{\Omega_W\cap T_\alpha}\widehat{\varphi}(z)\,d\mu_\xi(z)$$

for all $\varphi\in\mathscr{S}_\alpha(\mathbb{R}^n)$, proving theorem 5.12.

While we have proven theorem 5.12, we have also obtained more information on the special case of $\alpha=0$.

Corollary 6.10. *Let $W<\mathrm{O}(n)$ be finite. A tempered distribution $\xi\in\mathscr{S}(\mathbb{R}^n)$ is W-positive-definite if and only if it is W-invariant and positive-definite.*

Proof. By corollary 5.13 and lemma 6.9, the measure μ_ξ on $\mathbb{R}^n$ is tempered. Therefore, the W-invariant distribution $\xi=\widehat{\mu}_\xi$ is positive-definite by the Bochner-Schwartz theorem. ■

For example, recall that the Krein measure μ_n of the (evenly) positive-definite distribution $\xi_n = \mathscr{A}^{-1}\delta_{\mathbb{Z}^n}$ is an even tempered measure. Corollary 6.10 tells us that this is no coincidence.

7 The Abel Transform on Symmetric Spaces

To conclude this paper, we construct evenly positive-definite distributions on $\mathbb{R}$ using the inverse Abel transform, this time with the underlying geometry being that of the real and complex hyperbolic plane. To define the Abel transform, we will need some background on Lie groups. We take inspiration from S. Pusti's article [15] and some basic information on semisimple Lie groups can be found in [13].

Let G be a non-compact semisimple Lie group with finite center and $K<G$ a maximal compact subgroup. Our main examples of this will be $G=\mathrm{SL}_2(\mathbb{R}),\ \mathrm{SL}_2(\mathbb{C})$ and $K=\mathrm{SO}(2),\ \mathrm{SU}(2)$. On such a group G there is the notion of an *Iwasawa decomposition*, i.e. a diffeomorphism of multiplication

$$A\times N\times K\longrightarrow G$$

where A is a maximal torus called the *Cartan subgroup* and N is a maximal unipotent subgroup. On A we have an action of the *Weyl group* $W = N_G(A)/Z_G(A)$ acting by conjugation, where $N_G(A)$ is the normalizer of A in G and $Z_G(A)$ is the centralizer of A in G. One can show that W is finite and that W also acts on elements of the Lie algebra $\mathfrak{a} \cong A$ by reflections when endowed with the Killing form as the inner product, so $W < \mathrm{O}(\mathfrak{a})$. On G, there is also a *Cartan decomposition*, a diffeomorphism of multiplication

$$K \times A^+ \times K \longrightarrow G,$$

where A^+ is a fundamental domain of $W\backslash A$ in A. In particular, we have a diffeomorphism

$$K\backslash G/K \cong W\backslash A.$$

To define the Abel transform on $X = G/K$ we need to pick a suitable measure on G. It is a locally compact group that we are dealing with, so there is a Haar measure on G, i.e. a positive G-invariant Radon measure. Let Δ_{AN} be the modular function of $AN < G$ and define $\rho = \sqrt{\Delta_{AN}}\big|_A$. One can then show that, when extended canonically to a homomorphism on all of G, ρ^2 is the modular function on G and

$$m_G = \rho^2 m_A \otimes m_N \otimes m_K.$$

Regarding functions, if F is a sheaf of functions on G, for example C_c or C_c^∞, then we define for each open $U \subset G$ the subset $F(U,K) \subset F(U)$ of bi-K-invariant elements. A function on X is said to be *radial* if it is right-K-invariant, or equivalently, a bi-K-invariant function on G. We now have all the machinery to define the Abel transform on $X = G/K$.

Definition 7.1. *Let $f \in C_c(G,K)$. The Abel transform on G is the map $\mathscr{A} : C_c(G,K) \to C_c(A)$ given by*

$$\mathscr{A}f(a) = \rho(a)\int_N f(an)\,dm_N(n).$$

Remark. The natural question regarding this definition is what connection this has to the Euclidean case. To see this, consider the isometry group $G = \mathrm{O}(n) \ltimes \mathbb{R}^n$ and the compact subgroup $K = \mathrm{O}(n)$ so that $\mathbb{R}^n = G/K$. We can decompose G as ANK for $A = \mathbb{R}$, $N = A^\perp = \mathbb{R}^{n-1}$ in $\mathbb{R}^n$ and the Abel transform can for a radial function $f \in C_c(G,K)$ be written as

$$\mathscr{A}f(a) = \int_N f(a+n)\,dm_N(n).$$

The group G is in this case unimodular, so we do not have to worry about scaling using the modular function. What we really are using in the case of a Lie group G is the Iwasawa decomposition, so more generally we can define the Abel transform on any Gelfand pair (G,K) admitting a decomposition $G = ANK$ with A being abelian.

Lemma 7.2. *The Abel transform $\mathscr{A} : C_c(G,K) \to C_c(A)$ is a $*$-homomorphism.*

Proof. Let $f,h \in C_c(G,K)$ and denote the Haar measures on A,N,K and G by da,dn,dk and dg respectively. Then, using that f,h are bi-K-invariant we have

$$\begin{aligned}
\mathscr{A}(f*h)(a) &= \rho(a)\int_N (f*h)(an)\,dn \\
&= \rho(a)\int_N\int_G f(g^{-1}an)h(g)\,dg\,dn \\
&= \rho(a)\int_N\int_K\int_A\int_N f(m^{-1}b^{-1}k^{-1}an)h(kbm)\,dm\,db\,dk\,dn \\
&= \rho(a)\int_N\int_A\int_N f(m^{-1}b^{-1}an)h(bm)\,dm\,dn\,db \\
&= \rho(a)\int_A\int_N\int_N f(b^{-1}am^{-1}n)h(bm)\,dn\,dm\,db
\end{aligned}$$

$$
\begin{aligned}
&= \int_A \int_N \rho(b)\Big(\rho(b^{-1}a)\int_N f(b^{-1}am^{-1}n)dn\Big)h(bm)dmdb \\
&= \int_A \mathscr{A}f(b^{-1}a)\Big(\rho(b)\int_N h(bm)dm\Big)db \\
&= \int_A \mathscr{A}f(b^{-1}a)\mathscr{A}h(b)db = (\mathscr{A}f * \mathscr{A}h)(a).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\mathscr{A}f^*(a) &= \rho(a)\int_N \overline{f(n^{-1}a^{-1})}dn \\
&= \rho(a)\rho(a^{-1})^2\int_N \overline{f(a^{-1}n^{-1})}dn \\
&= \rho(a^{-1})\int_N \overline{f(a^{-1}n)}dn \\
&= \overline{\mathscr{A}f(a^{-1})} = (\mathscr{A}f)^*(a).
\end{aligned}
$$

■

One can, as shown in [13], that the action of the Weyl group W leaves $\mathscr{A}f$ fixed, so the Abel transform is a map $\mathscr{A} : C_c(G,K) \to C_c(A)^W$. Next we consider the geometry of the hyperbolic plane, and W-invariance of $\mathscr{A}$ will be shown through calculations.

7.1 The Hyperbolic Plane

Let us compute the inverse Abel transform for an example of a pair (G,K). Two of the simplest non-compact semisimple Lie groups are $G = \mathrm{SL}_2(\mathbb{R})$ and $G = \mathrm{SL}_2(\mathbb{C})$, and for each of them we pick the maximal compact subgroup $K = \mathrm{SO}(2)$ and $K = \mathrm{SU}(2)$ respectively. The corresponding homogeneous spaces

$$
\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \quad \text{and} \quad \mathbb{H}^2_{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)
$$

are called the *real* and *complex hyperbolic plane*. One can model the geometry of these spaces as 2- and 3-dimensional hyperbolic space. Let us do it for the real hyperbolic plane. Consider the upper half plane of complex numbers $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ and endow it with the metric topology of

$$
d(z,w) = \operatorname{arccosh}\Big(1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\Big).
$$

Let $G = \mathrm{SL}_2(\mathbb{R})$ act on this space by Möbius transformations, that is,

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}, \quad ad - bc = 1.
$$

It can be shown that this action is smooth, transitive and that $d(g.z, g.w) = d(z,w)$ for all $g \in G$ and z,w in the upper half plane. Note that if

$$
g.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = z \quad \text{then} \quad (b+c) + i(a-d) = 0,
$$

which holds if and only if $g \in \mathrm{SO}(2)$, so the upper half plane in this metric topology is actually diffeomorphic to the real hyperbolic plane as we have defined it. Similarly, one can show that the complex upper half plane $\{(z,t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$ is diffeomorphic to the complex hyperbolic plane. These spaces are examples of so called *Riemannian symmetric spaces*. However, to compute the inverse Abel transform on these spaces, we will only need the structure of the groups $G = \mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{C})$ as Lie groups.

Let us write out the objects discussed in the beginning of section 7. In the real case, the Iwasawa decomposition of G is given by

$$K=\left\{k_\theta=\begin{pmatrix}\cos(2\pi\theta) & \sin(2\pi\theta)\\ -\sin(2\pi\theta) & \cos(2\pi\theta)\end{pmatrix}:\theta\in[0,1)\right\}\cong S^1$$
$$A=\left\{a_t=\begin{pmatrix}e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{pmatrix}:t\in\mathbb{R}\right\}\cong\mathbb{R}$$
$$N=\left\{n_s=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}:s\in\mathbb{R}\right\}\cong\mathbb{R},$$

and the Weyl group of G is $W=\mathrm{O}(1)$ on A, sending $a_t\mapsto a_{\pm t}$. From this one can show that there are homeomorphisms

$$W\backslash A\xrightarrow{\iota_A}[1,\infty)\xleftarrow{\iota}K\backslash G/K$$

sending a matrix g in either quotient space to $\frac{1}{2}\mathrm{tr}(g^*g)$. Note that $\frac{1}{2}\mathrm{tr}(a_t^*a_t)=\cosh(t)$. These maps dualize to $*$-algebra isomorphisms

$$C_c(A)^W\xleftarrow{\iota_A^*}C_c[1,\infty)\xrightarrow{\iota^*}C_c(G,K)$$

and we introduce the transform $\mathscr{A}_\mathbb{R}:C_c[1,\infty)\longrightarrow C_c[1,\infty)$ by

$$\mathscr{A}_\mathbb{R}F(x)=\int_\mathbb{R}F(x+\frac{s^2}{2})\,ds\,.$$

To determine the Abel transform, we need to determine the character ρ. On readily checks that $a_tn_s=n_{e^ts}a_t$, so for any integrable function f on G,

$$\int_\mathbb{R}f(a_tn_s)\,ds=e^{-t}\int_\mathbb{R}f(n_sa_t)\,ds\,.$$

Thus we take $\rho(a_t)=e^{t/2}$. As $\iota(a_tn_s)=\frac{1}{2}\mathrm{tr}(n_s^*a_{2t}n_s)=\cosh(t)+\frac{1}{2}s^2e^t$, then for $f=\iota^*F$ we have that

$$\mathscr{A}f(a_t)=e^{t/2}\int_\mathbb{R}F(\cosh(t)+\frac{1}{2}s^2e^t)\,ds=\int_\mathbb{R}F(\cosh(t)+\frac{u^2}{2})\,du=\iota_A^*\mathscr{A}_\mathbb{R}F(t)\,.$$

In summary we have a commuting diagram

$$\begin{array}{ccc}C_c(G,K) & \xrightarrow{\mathscr{A}} & C_c(A)^W\\ \iota^*\uparrow & & \uparrow\iota_A^*\\ C_c[1,\infty) & \xrightarrow{\mathscr{A}_\mathbb{R}} & C_c[1,\infty)\,.\end{array}$$

In the complex case, when the underlying field of G is $\mathbb{C}$, then

$$K=\left\{k_{\alpha,\beta}=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}:|\alpha|^2+|\beta|^2=1,\alpha,\beta\in\mathbb{C}\right\}\cong S^3$$
$$A=\left\{a_t=\begin{pmatrix}e^{\frac{t}{2}} & 0\\ 0 & e^{-\frac{t}{2}}\end{pmatrix}:t\in\mathbb{R}\right\}\cong\mathbb{R}$$
$$N=\left\{n_z=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}:z\in\mathbb{C}\right\}\cong\mathbb{C},$$

and $W=\mathrm{O}(1)$. Since the maximal torus A is the same as before, the maps ι,ι_A and their duals remain the same. This time, consider the Volterra-type operator $\mathscr{A}_\mathbb{C}:C_c[1,\infty)\longrightarrow C_c[1,\infty)$ given by

$$\mathscr{A}_\mathbb{C}F(x)=2\pi\int_0^\infty F(x+s)\,ds\,.$$

Regarding the modular function on AN, we again have $a_t n_z = n_{e^t z} a_t$ and by the same reasoning as in the real case we take $\rho(a_t) = e^t$, accounting for the two real dimensions of $\mathbb{C}$. We have $\iota(a_t n_z) = \frac{1}{2}\mathrm{tr}(n_z^* a_{2t} n_z) = \cosh(t) + \frac{1}{2}|z|^2 e^t$ and so for $f = \iota^* F$,

$$\begin{aligned}\mathscr{A} f(a_t) &= e^t \int_{\mathbb{C}} F(\cosh(t) + \frac{1}{2}|z|^2 e^t)\, dz \\ &= 2\pi \int_0^\infty F(\cosh(t) + \frac{r^2}{2})\, r\, dr \\ &= 2\pi \int_0^\infty F(\cosh(t) + r)\, dr = \iota_A^* \mathscr{A}_{\mathbb{C}} F(t).\end{aligned}$$

Again we have a commuting square

$$\begin{array}{ccc} C_c(G,K) & \xrightarrow{\mathscr{A}} & C_c(A)^W \\ {\scriptstyle \iota^*}\uparrow & & \uparrow{\scriptstyle \iota_A^*} \\ C_c[1,\infty) & \xrightarrow{\mathscr{A}_{\mathbb{C}}} & C_c[1,\infty)\,. \end{array}$$

Lemma 7.3. *The maps* $\mathscr{A}_{\mathbb{R}}, \mathscr{A}_{\mathbb{C}} : C_c^\infty[1,\infty) \to C_c^\infty[1,\infty)$ *are linear isomorphisms with inverses given by*

$$\begin{aligned}\mathscr{A}_{\mathbb{R}}^{-1} F(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} F'(x + \frac{u^2}{2})\, du\,, \\ \mathscr{A}_{\mathbb{C}}^{-1} F(x) &= -\frac{1}{2\pi} F'(x).\end{aligned}$$

Here we view $C_c^\infty[1,\infty)$ as compactly supported smooth *even* functions on the multiplicative group $\mathbb{R}^+ = (0,\infty)$.

Proof. We need to show that $\mathscr{A}\mathscr{A}^{-1} = 1$ and that $\mathscr{A}^{-1}$ is bounded in the Frechét topology on $C_c^\infty[1,\infty)$. For $\mathscr{A}_{\mathbb{C}}$ this follows from the fundamental theorem of calculus. For $\mathscr{A}_{\mathbb{R}}$ however, we have

$$\begin{aligned}\mathscr{A}_{\mathbb{R}} \mathscr{A}_{\mathbb{R}}^{-1} F(x) &= -\frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} F'(x + \frac{s^2}{2} + \frac{u^2}{2})\, du\, ds \\ &= -\int_0^\infty F'(x + \frac{r^2}{2}) r\, dr \\ &= -\int_0^\infty F'(x + v)\, dv = F(x).\end{aligned}$$

For boundedness of $\mathscr{A}_{\mathbb{R}}^{-1}$, we refer to lemma 5.5 in [BHPII], where it is proven that

$$\left\| \mathscr{A}_{\mathbb{R}}^{-1} F^{(m)} \right\|_\infty \leq \frac{2\sqrt{2}}{\pi} \max\left(\left\| F^{(m)} \right\|_\infty, \left\| F^{(m+1)} \right\|_\infty \right).$$

■

If $\varphi = \iota_A^* F \in C_c^\infty(A)^W \cong C_c^\infty(\mathbb{R})^{\mathrm{even}}$, then

$$F'(x) = \frac{d}{dx}((\iota_A^*)^{-1}\varphi)(x) = \frac{\varphi'(\pm \mathrm{arcosh}(x))}{\sqrt{x^2 - 1}}$$

where φ' is the derivative of φ as a function on $\mathbb{R}$. Recall that we have a diffeomorphism $\iota^{-1} \circ \iota_A : W\backslash A \to K\backslash G/K$, so for every double coset $KgK \in K\backslash G/K$ there is an $x = \frac{1}{2}\mathrm{tr}(g^* g) \geq 1$ such that

$KgK = Ka_{\mathrm{arcosh}(x)}K$. Thus for such an x,

$$\mathscr{A}^{-1}\varphi(g) = -\frac{1}{\pi}\int_0^\infty \frac{\varphi'(\pm\mathrm{arcosh}(x+r^2/2))}{\sqrt{(x+r^2/2)^2-1}}\,dr$$

in the real case and

$$\mathscr{A}^{-1}\varphi(g) = -\frac{\varphi'(\pm\mathrm{arcosh}(x))}{2\pi\sqrt{x^2-1}}$$

in the complex case. Together with lemma 6.2 and the $*$-isomorphisms ι^*, ι_A^*, we have that the Abel transforms, concretely interpreted as

$$\mathscr{A} : C_c^\infty(\mathbb{H}^2)^{\mathrm{SO}(2)} \longrightarrow C_c^\infty(\mathbb{R})^{\mathrm{even}} \quad \text{and} \quad \mathscr{A} : C_c^\infty(\mathbb{H}^2_{\mathbb{C}})^{\mathrm{SU}(2)} \longrightarrow C_c^\infty(\mathbb{R})^{\mathrm{even}},$$

are $*$-isomorphisms. Note that as $\mathscr{A}_{\mathbb{R}}$ is non-local and $\mathscr{A}_{\mathbb{C}}$ is local then the corresponding Abel transforms on $\mathbb{H}^2$ and $\mathbb{H}^2_{\mathbb{C}}$ will be non-local and local respectively. By lemma 6.3 it makes sense to talk about the dual map $\mathscr{A}^{-1} : C_c^\infty(G,K)^* \to (C_c^\infty(A)^W)^*$ on distributions. Due to the exponential growth of balls in the hyperbolic plane, a distribution $\xi = \mathscr{A}^{-1}\eta$, $\eta \in C_c^\infty(G,K)^*$ is not necessarily a tempered distribution on $A = \mathbb{R}$. Moreover, the inverse Abel transform on the real hyperbolic plane $\mathbb{H}^2$ is not local, so it does not preserve pure point distributions in general. The inverse Abel transform on the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$ however, does, due to locality. This preserves the idea of locality in odd dimensions as in the Euclidean case. Let's compute some examples:

Take the lattice $\Gamma = \mathrm{SL}_2(\mathbb{Z}) < G = \mathrm{SL}_2(\mathbb{R})$ and consider the distribution $\xi_\Gamma = \mathscr{A}^{-1}\delta_\Gamma$. Define for each $m \in \mathbb{N}$,

$$r_\Gamma(m) = \left|\left\{\gamma \in \Gamma : m = \frac{1}{2}\mathrm{tr}(\gamma^*\gamma)\right\}\right| = \left|\left\{(a,b,c,d) \in \mathbb{Z}^4 : \begin{cases} a^2+b^2+c^2+d^2 = 2m \\ ad-bc = 1 \end{cases}\right\}\right|.$$

Then the distribution ξ_Γ takes the form

$$\xi_\Gamma(\varphi) = -\frac{1}{\pi}\sum_{m=1}^\infty r_\Gamma(m)\int_0^\infty \frac{\varphi'(\mathrm{arcosh}(m+r^2/2))}{\sqrt{(m+r^2/2)^2-1}}\,dr.$$

Similarly, we can consider $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i]) < G = \mathrm{SL}_2(\mathbb{C})$ and the distribution $\xi_\Gamma = \mathscr{A}^{-1}\delta_\Gamma$. To write it down we need to determine $\mathscr{A}^{-1}\varphi$ for $t = 0$, which by l'Hopital's rule is

$$\lim_{x\to 1}\frac{\varphi'(\mathrm{arcosh}(x))}{\sqrt{x^2-1}} = \lim_{x\to 1}\frac{\varphi''(\mathrm{arcosh}(x))}{x} = \varphi''(0).$$

Define the coefficients

$$r_\Gamma(m) = \left|\left\{\gamma \in \Gamma : m = \frac{1}{2}\mathrm{tr}(\gamma^*\gamma)\right\}\right| = \left|\left\{(a,b,c,d) \in \mathbb{Z}[i]^4 : \begin{cases} |a|^2+|b|^2+|c|^2+|d|^2 = 2m \\ ad-bc = 1 \end{cases}\right\}\right|.$$

One notes that $r_\Gamma(1) = 8$ using different combinations of $1, -1, i, -i$, so finally we have

$$\xi_\Gamma(\varphi) = -\frac{4}{\pi}\varphi''(0) - \frac{1}{2\pi}\sum_{m=2}^\infty \frac{r_\Gamma(m)}{\sqrt{m^2-1}}\varphi'(\mathrm{arcosh}(m)),$$

which can be seen as a hyperbolic analouge of Guinand's distribution. By the Gelfand-Vilenkin-Krein theorem there is a Krein measure μ_Γ on $\mathbb{R} \cup i\mathbb{R} \subset \mathbb{C}$ such that $\xi_\Gamma = \widehat{\mu}_\Gamma$. What this measure actually is can be answered through the theory of *spherical diffraction*, which was one of the motivating areas for this paper. One can show that the support and densities of μ_ξ are determined by the irreducible K-spherical representations of G, and the representations parametrized by the real and imaginary axes will correspond to the principal and complementary series ones. The measure μ_Γ is *not* purely supported on $\mathbb{R} \subset \mathbb{C}$ and so ξ_Γ is not a tempered distribution by corollary

5.13.

A Some Algebraic Geometry

This section regards the proof of theorem 6.2, which draws ideas from basic algebraic geometry. For an introduction on the subject, see [8].

Let k be a field, for example $\mathbb{R}$ or $\mathbb{C}$. Given a polynomial $F \in k[X_1,...,X_n]$, we denote by $V(F)$ it's zero set and more generally $V(F_1,...,F_m)$ for the common zero set of polynomials $F_1,...,F_m$. If we denote by J the ideal generated by $F_1,...,F_m$ in $k[X_1,...,X_n]$ then $V(J) = V(F_1,...,F_m)$, so every so called *algebraic set* in k^n is of the form $V(J)$ for some ideal $J \subset k[X_1,...,X_n]$. Every such ideal is moreover finitely generated by the following theorem:

Theorem A.1. *(Hilbert Basis Theorem) If R is a Noetherian ring, then $R[X_1,...,X_n]$ is Noetherian. In particular, every ideal in $k[X_1,...,X_n]$ is finitely generated.*

A minimal algebraic set is called a *variety*. Conversely to the ideals, for each subset $V \subset k^n$ we define $I(V) \subset k[X_1,...,X_n]$ to be the ideal generated by all polynomials whose zero set contains V. By the Hilbert basis theorem, $I(V)$ is generated by a finite set of polynomials. A fundamental result regarding algebraic sets in k^n and ideals in $k[X_1,...,X_n]$ is the so called *Hilbert's Nullstellensatz*. In order to state it we define the *radical* of an ideal $J \subset k[X_1,...,X_n]$ to be the ideal

$$\mathrm{Rad}(J) = \left\{F \in k[X_1,...,X_n] : F^m \in J \text{ for some } m \geq 0\right\}.$$

Theorem A.2. *(Hilbert's Nullstellensatz) Let k be an algebraically closed field and let $J \subset k[X_1,...,X_n]$ be an ideal. Then $I(V(J)) = \mathrm{Rad}(J)$.*

For a proof, see [8]. The following corollary strengthens the relation between algebraic sets in k^n and ideals in $k[X_1,...,X_n]$.

Corollary A.3. *(Weak Nullstellensatz) Let k be an algebraically closed field. If $J \subset k[X_1,...,X_n]$ is a proper ideal, then $V(J) \neq \varnothing$.*

Proof. Suppose that $V(J) = \varnothing$. Then the ideal $I(V(J))$ contains a polynomial with empty zero set and since k is algebraically closed, it must be a nonzero constant polynomial. Thus $1 \in I(V(J)) = \mathrm{Rad}(J)$, meaning that $1 = 1^m \in J$ for some $m \geq 0$. But then $J = k[X_1,...,X_n]$. ■

We are now ready to prove Theorem 6.2, and we divide it into a lemma and two theorems.

Lemma A.4. *(Hilbert's 14th Problem) Let $W < \mathrm{O}(n)$ be a closed subgroup. Then the ring $\mathbb{C}[X_1,...,X_n]^W$ is finitely generated.*

Proof. Let $J \subset \mathbb{C}[X_1,...,X_n]$ be the ideal generated by all homogeneous W-invariant polynomials of positive degree. Then by the Hilbert basis theorem, J is finitely generated. Each generator is contained in an ideal generated by homogeoneous W-invariant polynomials of positive degree, so without loss of generality, J is generated by finitely many such polynomials $F_1,...,F_r$.

Now suppose that $h \in \mathbb{C}[X_1,...,X_n]^W$ is homogenenous of degree d. We prove that $h \in \mathbb{C}[F_1,...,F_m]$ using induction on d. The case $d = 0$ is trivial. If $d > 0$ then there are homogenous polynomials $a_1,...,a_m \in \mathbb{C}[X_1,...,X_n]$ such that $\deg(a_i) = d - \deg(F_i)$ and

$$h = \sum_{i=1}^{m} a_i F_i.$$

Now, as W is compact then there is a Haar probability measure μ_W on W, and we consider the averaging operation

$$F^W = \int_W w.F \, d\mu_W(w),$$

where $w.F(X) = F(w.X)$. Since h and each F_i is W-invariant then

$$h = h^W = \sum_{i=1}^{m} a_i^W F_i$$

and it is clear that a_i^W remains homogeneous of degree $d - \deg(F_i) < d$. Since a_i^W is W-invariant then $a_i^W \in \mathbb{C}[F_1,...,F_m]$ by the induction assumption and finally we have $h \in \mathbb{C}[F_1,...,F_m]$. For a general $h \in \mathbb{C}[X_1,...,X_n]^W$ of degree d we can write $h = h_0 + ... + h_d$, where h_i is a homogeneous polynomial of degree i. Then $h_i \in \mathbb{C}[F_1,...,F_m]$ for all i and so $h \in \mathbb{C}[F_1,...,F_m]$. ■

Theorem A.5. *Let $W < \mathrm{O}(n)$ be closed. Then every algebra homomorphism $\mathbb{C}[X_1,...,X_n]^W \longrightarrow \mathbb{C}$ is an evaluation at some $z \in \mathbb{C}^n$.*

Proof. Suppose that $\lambda : \mathbb{C}[X_1,...,X_n]^W \longrightarrow \mathbb{C}$ is an algebra homomorphism. By lemma A.4 we can find finitely many generators $F_1,...,F_m$ of $\mathbb{C}[X_1,...,X_n]^W$ and so λ is uniquely determined by the vector $\beta = (\lambda(F_1),...,\lambda(F_m)) \in \mathbb{C}^m$. Consider the ideal

$$I_\beta = (F_1 - \beta_1,...,F_m - \beta_m)$$

with the corresponding variety $V_\beta = V(I_\beta)$. The polynomials $F_i - \beta_i$ are clearly W-invariant, so I_β must be a proper ideal in $\mathbb{C}[X_1,...,X_n]$. By the Hilbert Nullstellensatz $V_\beta \neq \varnothing$ and for any $z \in V_\beta$, $\lambda(F_i) = F_i(z)$ for all i, which means that λ is evaluation at $z \in V_\beta$.

■

Theorem A.6. *Let $W < \mathrm{O}(n)$ be finite. Then every evaluation map $\mathbb{C}[X_1,...,X_n]^W \longrightarrow \mathbb{C}$ is unique, up to the action of W.*

Proof. To prove uniquness, it suffices to show that there is a W-invariant polynomial $F \in \mathbb{C}[X_1,...,X_n]$ that separates W-orbits. As W acts on $\mathbb{C}^n$ by linear maps and the W-orbits are finite, they define complex varieties. Therefore, if $z, z' \in \mathbb{C}^n$ lie in different W-orbits then there are polynomials $F_1,...,F_k$ and $F'_1,...,F'_l$ in $\mathbb{C}[X_1,...,X_n]$ such that

$$W.z = V(F_1,...,F_k)\,, \quad W.z' = V(F'_1,...,F'_l)$$

and since orbits are disjoint then F_i, F'_j have no common zeroes for all i,j. This means that $V(F_1,...,F'_l) = W.z \cap W.z' = \varnothing$ and so by the Hilbert Nullstellensatz $1 \in (F_1,...,F'_l)$, which means that there are $a_1,...,a_k,a'_1,...,a'_l$ in $\mathbb{C}[X_1,...,X_n]$ such that

$$1 = \sum_i a_i F_i + \sum_j a'_j F'_j.$$

Let $h_1 = \sum_i a_i F_i$ and $h_2 = \sum_j a'_j F'_j$ so that $1 = h_1 + h_2$. If we make this W-invariant then $h_1^W + h_2^W = 1$ and by definition we have $h_1^W|_{W.z} = 0 = h_2^W|_{W.z'}$. Thus if we take $F = h_1^W$ then $F|_{W.z} = 0$ and $F|_{W.z'} = (1 - h_2)|_{W.z'} = 1$.

■

Next we highlight an example of a non-finite W for which evaluation maps $\mathbb{C}[X_1,...,X_n]^W \longrightarrow \mathbb{C}$ are *not* unique up to the action of W.

Given a variety $V \subset k^n$ we define the *Zariski topology* on V to be the topology generated by the algebraic subsets of V as the closed sets. If $k = \mathbb{R}$ then we define the *complexification* of a variety $V \subset \mathbb{R}^n$ to be the set of $z \in \mathbb{C}^n$ such that $F(z) = 0$ for all $F \in I(V)$. Moreover, to a variety V we define it's *ring of polynomials* by

$$\mathscr{O}(V) = k[X_1,...,X_n]/I(V).$$

The ring of polynomials on the complexification of a real variety turns out to be closely related to the initial ring of polynomials.

Lemma A.7. *Let V be a real variety and $V_{\mathbb{C}}$ its complexification. If $f \in \mathscr{O}(V_{\mathbb{C}})$ and $f = 0$ on V then $f = 0$ on $V_{\mathbb{C}}$. In other words, V is Zariski-dense in $V_{\mathbb{C}}$.*

Proof. It suffices to show that $\mathscr{O}(V_{\mathbb{C}}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathscr{O}(V)$, and since $\mathbb{C}[X_1,\dots,X_n] \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X_1,\dots,X_n]$ it then suffices to show $I(V_{\mathbb{C}}) \cong \mathbb{C} \otimes_{\mathbb{R}} I(V)$. Clearly the RHS is contained in the LHS, and since $I(V_{\mathbb{C}})$ is radical then it remains to show that $\mathbb{C} \otimes_{\mathbb{R}} I(V)$ is a radical ideal in $\mathbb{C}[X_1,\dots,X_n]$.

Suppose $p, q \in \mathbb{R}[X_1,\dots,X_n]$ and that $(p+iq)^m \in \mathbb{C} \otimes_{\mathbb{R}} I(V)$ for some $m \geq 0$. Then

$$(p^2+q^2)^m = (p-iq)^m (p+iq)^m \in \mathbb{C} \otimes_{\mathbb{R}} I(V),$$

and since $I(V)$ is a radical ideal then $p^2+q^2 \in I(V)$. Thus $p^2+q^2 = 0$ on V and so $p = q = 0$ on V, meaning that $p, q \in I(V)$ and finally that $p+iq \in \mathbb{C} \otimes_{\mathbb{R}} I(V)$. ∎

Consider $W = \mathrm{O}(3)$, acting transitively on the 5-sphere S^5 in $\mathbb{C}^3 = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^3$ and take the points $z = (1,0,0)$, $z' = (i,0,0)$. Then $W.z$ and $W.z'$ are two disjoint copies of the 2-sphere S^2 inside S^5, yet they cannot be separated by a *complex* polynomial on $\mathbb{C}^3$ by lemma A.5. Thus theorem A.4 does not hold for the evaluation map $F \mapsto F(z) = F(z')$.

B The Plancherel-Godement Theorem

We shall state and give a proof of an abstract version of the Plancherel-Godement theorem for positive functionals on complex involutive algebras. Throughout this section, A will denote a commutative topological $*$-algebra over the complex numbers and $\alpha \in A^*$ a $*$-positive functional in the sense that

$$\alpha(xx^*) \geq 0$$

for all $x \in A$. The main idea of the Plancherel-Godement theorem will be to express α in terms of a positive Radon measure ν_α on some locally compact space Ω_α.

We will be interested in under what conditions we can extend α to the unitalization of A. We define the unitalization A^+ of A to be the complex vector space

$$A^+ = A \oplus \mathbb{C},$$

endowed with the operation $(x_1,\lambda_1)(x_2,\lambda_2) = (x_1x_2 + \lambda_2 x_1 + \lambda_1 x_2, \lambda_1\lambda_2)$ and the involution $(x,\lambda)^* = (x^*, \overline{\lambda})$, and we write $x+\lambda$ in place of (x,λ). It is clear that A^+ defines a complex involutive commutative algebra.

Given an algebra A as described, we can define the *Gelfand transform* to interpret A in terms of a commutative C^*-algebra. First we need the notion of a *spectrum*. Endow A^* with the coarsest topology of pointwise convergence and define the *spectrum of characters* on A by

$$\operatorname{Spec} A = \left\{ \chi \in A^* \setminus \{0\} : \chi(ab^*) = \chi(a)\overline{\chi(b)} \ \forall a, b \in A \right\},$$

i.e. the subspace of A^* consisting of nonzero, complex-valued $*$-homomorphisms. If A is a commutative Banach algebra, then there is an identification of $\operatorname{Spec} A$ with the maximal ideal space of A as a ring by sending $\chi \in \operatorname{Spec} A$ to its kernel. Moreover, for such an A, $\operatorname{Spec} A$ is a locally compact subspace of A^* by the Banach-Alaoglu theorem and if A is unital then $\operatorname{Spec} A$ is closed, hence compact. The Gelfand transform on A is the $*$-homomorphism

$$\Gamma : A \longrightarrow C_0(\operatorname{Spec} A),$$

sending $x \in A$ to the evaluation map $\widehat{x} : \chi \mapsto \chi(x)$.

A notion we will need when studying extensions of α to A^+ is that of a *positive-definite element* of A.

Definition B.1. *Let A and α be as above. An element $p \in A$ is positive-definite with respect to α if the functional*

$$\alpha_p(x) = \alpha(px)$$

extends to a $*$-positive functional on A^+.

Example. If $x \in A$ then $xx^* \in A$ is positive-definite with respect to α , for if we define

$$\alpha_{xx^*}(y+\lambda) = \alpha(xx^*y + \lambda xx^*)$$

then

$$\alpha_{xx^*}((y+\lambda)(y+\lambda)^*) = \alpha((xy+\lambda x)(xy+\lambda x)^*) \geq 0\,.$$

The Plancherel-Godement theorem can now be stated in terms of positive-definite elements in A.

Theorem B.2. *(Plancherel-Godement) Let A be an algebra as above with the addition of having an approximate identity $\{a_U\}$, and let $\alpha \in A^*$ be a $*$-positive functional satisfying*

$$\alpha(xyy^*x^*) \leq k_y \alpha(xx^*)\,, \quad k_y \geq 0$$

for every $x, y \in A$. Moreover, denote by $\mathfrak{p}_A$ the dense ideal in A generated by positive-definite elements. Then there is a unique positive Radon measure ν_α on a locally compact subspace $\Omega_\alpha \subset$ Spec$A$ *such that for every $x \in \mathfrak{p}_A$,*

$$\alpha(x) = \int_{\Omega_\alpha} \widehat{x}\, d\nu_\alpha\,.$$

To prove the theorem we prove a Bochner theorem for functionals on commutative Banach algebras A that extend to A^+ and then obtain the measure ν_α on Spec$A$ by means of positive-definite elements. First we need to find a suitable criterion for when $*$-positive functionals on A extend to $*$-positive functionals on the unitalization of A.

In general, a $*$-positive functional $\alpha \in A^*$ defines a positive hermitian form h_α on A by $h_\alpha(x,y) = \alpha(xy^*)$. It satisfies the Cauchy-Schwarz inequality, which reads as

$$|\alpha(xy^*)|^2 \leq \alpha(xx^*)\alpha(yy^*)\,.$$

If A is unital then for $k = \alpha(1)$ we have

(1) $\alpha(x^*) = \overline{\alpha(x)}$, and

(2) $|\alpha(x)|^2 \leq k\alpha(xx^*)$.

It turns out that this is a sufficient criterion for α to be extendable to A^+.

Lemma B.3. *A $*$-positive functional $\alpha : A \to \mathbb{C}$ extends to a $*$-positive functional on A^+ if and only if the conditions (1) and (2) hold.*

Proof. The "only if" statement has been proven, so assume that (1) and (2) holds. Then if we for $\lambda \in \mathbb{C}$ define $\alpha(\lambda) = \lambda k$ we get that $\alpha(x) \geq -k\alpha(xx^*)$ and

$$\begin{aligned}
\alpha((x+\lambda)(x+\lambda)^*) &= \alpha(xx^*) + 2\mathrm{Re}(\overline{\lambda}\alpha(x)) + |\lambda|^2 k \\
&\geq \alpha(xx^*) - 2|\lambda|\sqrt{k\alpha(xx^*)} + |\lambda|^2 k \\
&= (\sqrt{\alpha(xx^*)} - |\lambda|\sqrt{k})^2 \geq 0\,.
\end{aligned}$$

∎

With this criterion we can state the Bochner theorem for C^*-algebras.

Theorem B.4. *(Bochner) Let A be a commutative C^*-algebra. Then a $*$-positive functional $\alpha \in A^*$ extends to a $*$-positive functional on A^+ if and only if there is a unique finite positive Borel measure μ_α on* $\operatorname{Spec} A$ *such that*

$$\alpha(x) = \int_{\operatorname{Spec} A} \widehat{x}\, d\mu_\alpha \,.$$

Proof. If μ_α is finite positive Borel measure such that $\alpha(x) = \int_{\operatorname{Spec} A} \widehat{x}\, d\mu_\alpha$ then

(1) $\alpha(xx^*) = \int |\widehat{x}|^2\, d\mu_\alpha \geq 0$,

(2) $\alpha(x^*) = \int \overline{\widehat{x}}\, d\mu_\alpha = \overline{\int \widehat{x}\, d\mu_\alpha} = \overline{\alpha(x)}$, and

(3) $|\alpha(x)|^2 = \left|\int \widehat{x}\, d\mu_\alpha\right|^2 \leq \|\mu_\alpha\| \int |\widehat{x}|^2\, d\mu_\alpha = \|\mu_\alpha\|\, \alpha(xx^*)$,

so α extends to a $*$-positive functional on A^+ by lemma B.3.

Conversely, if $\alpha \in A^*$ extends to a $*$-positive functional on A^+ then by iterating condition (2) we get that

$$\begin{aligned} |\alpha(x)|^2 \leq k\alpha(xx^*) &\leq k^{1+1/2}\alpha((xx^*)^2)^{1/2} \leq \ldots \leq k^{1+\ldots+2^{-n}}\alpha((xx^*)^{2^n})^{2^{-n}} \\ &\leq k^{1+\ldots+2^{-n}}\, \|\alpha\| \left\|(xx^*)^{2^n}\right\|^{2^{-n}} \underset{n}{\longrightarrow} k^2 r_A(xx^*), \end{aligned}$$

where r_A is the spectral radius on A. From the spectral theory of Banach algebras one can show that $r_A(x) = \sup_\chi |\chi(x)| = \|\widehat{x}\|_\infty$, so as $\chi(xx^*) = |\chi(x)|^2$ then $r_A(xx^*) = \|\widehat{x}\|_\infty^2$. Taking square roots we get that

$$|\alpha(x)| \leq k\, \|\widehat{x}\|_\infty \,,$$

meaning that we have a continuous functional μ_α on $\Gamma(A^+) \subset C(\operatorname{Spec} A^+)$ given by $\mu_\alpha(\widehat{x}) = \alpha(x)$. The subalgebra $\Gamma(A^+)$ is self-adjoint as A is and it separates points by definition, so as $\operatorname{Spec} A^+$ is compact then $\Gamma(A^+) \subset C(\operatorname{Spec} A^+)$ is dense by the Stone-Weierstrass theorem. By continuity, μ_α extends to a positive functional on $C(\operatorname{Spec} A^+)$ and by the Riesz representation theorem μ_α defines a unique finite positive Borel measure on $\operatorname{Spec} A^+$. Since α initially was defined on A, we may take μ_α restricted to $\operatorname{Spec} A$, and finally we have

$$\alpha(x) = \int_{\operatorname{Spec} A} \widehat{x}\, d\mu_\alpha \,.$$

∎

To prove the Plancherel-Godement theorem, we now construct a $*$-representation of A to lift the functional $\alpha \in A^*$ to a commutative C^*-algebra. On it we can make use of the Bochner theorem that we just proved.

By the Cauchy-Schwarz inequality,

$$|\alpha(xyy^*x^*)|^2 \leq \alpha(xyy^*yy^*x^*)\alpha(xx^*)$$

which means that the kernel $I_\alpha = \{x \in A : \alpha(xx^*) = 0\}$ is an ideal in A. If we denote by $\mathscr{H}_\alpha$ the Hilbert completion of the inner product space $(A/I_\alpha, h_\alpha)$, then right multiplication by $y \in A$ becomes a linear map $\pi(y) : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ and the requiremet that it be bounded is precisely that there is a constant k_y such that

$$\alpha(xyy^*x^*) \leq k_y \alpha(xx^*).$$

Therefore we obtain a $*$-representation

$$\pi : A \to B(\mathscr{H}_\alpha)$$

and the subalgebra $\pi(A) \subset B(\mathscr{H}_\alpha)$ is contained in a minimal self-adjoint, commutative C^*-algebra $B_\alpha \subset B(\mathscr{H}_\alpha)$, in which $\pi(A)$ is dense. We wish to lift α to a functional β on B_α, so that it remains to prove the theorem for commutative C^*-algebras. Define for each $x \in A$,

$$\beta(\pi(x)) = \alpha(x).$$

For β to define a functional on $\pi(A)$ we need to prove that $\alpha(x) = 0$ whenever $\pi(x) = 0$. First note that the kernel of π is

$$\ker \pi = \{x \in A : xy \in I_\alpha \ \forall y \in A\} = \{x \in A : \alpha(xyy^*x^*) = 0 \ \forall y \in A\}$$

and as $\alpha(xyy^*x^*) \le k_y \alpha(xx^*)$ for all $y \in A$ then

$$\{x \in A : \alpha(xyy^*x^*) = 0 \ \forall y \in A\} \supset \{x \in A : \alpha(xx^*) = 0\} = I_\alpha.$$

Replacing $y \in A$ with the approximate identity $\{a_U\}$ we see that if $x \in \ker \pi$ then $0 = \alpha(xa_U a_U^* x^*) \to \alpha(xx^*)$, so $\alpha(xx^*) = 0$. Thus we have shown $\ker \pi = I_\alpha$. This means that in order for β to define a functional $\pi(A) \to \mathbb{C}$, it suffices to show that $\alpha(x) = 0$ whenever $x \in I_\alpha$. By the Cauchy-Schwarz inequality,

$$|\alpha(xy^*)|^2 \le \alpha(xx^*)\alpha(yy^*),$$

so $\alpha(xy^*) = 0$ for all $y \in A$ whenever $x \in I_\alpha$. Taking $\{a_U\}$ in place of y, then $\alpha(x) = 0$ in the limit.

To prove continuity of the now well-defined functional $\beta : \pi(A) \to \mathbb{C}$, note that by the first isomorphism theorem,

$$\begin{array}{ccc} A & \xrightarrow{\ \pi\ } & \pi(A) \\ {\scriptstyle q_\alpha}\downarrow & \nearrow_{\cong} & \\ A/I_\alpha & & \end{array} \tag{B.1}$$

commutes. This means that π is an identification map in the topological sense, so the continuity of $\alpha = \beta \circ \pi$ is equivalent to the continuity of β. Now $\beta : \pi(A) \to \mathbb{C}$ is bounded and so it extends to all of B_α as $\pi(A)$ is dense.

From this construction it suffices to prove the Plancherel-Godement theorem for the algebra B_α, so without loss of generality, A is a commutative C^*-algebra.

B.1 Proof of the Plancherel-Godement Theorem for commutative C^*-algebras

Let A be a commutative C^*-algebra. If $p \in A$ is positive-definite, then by the Bochner theorem there is a unique finite positive Borel measure μ_p on $\operatorname{Spec} A$ such that

$$\alpha_p(x) = \int_{\operatorname{Spec} A} \widehat{x}\, d\mu_p.$$

Moreover, if $p, q \in A$ both are positive-definite then $\mu_p(\widehat{x}\widehat{q}) = \alpha(pqx) = \mu_q(\widehat{x}\widehat{p})$, which by continuity extends to

$$\mu_p(f\widehat{q}) = \mu_q(f\widehat{p})$$

for all $f \in C_b(\operatorname{Spec} A)$. To define a functional that takes $f \in C_0(\operatorname{Spec} A)$, we need the following lemma:

Lemma B.5. *For every compact subset $K \subset \operatorname{Spec} A$ there is a positive-definite $p \in A$ such that $\widehat{p}$ is strictly nonzero on K.*

Proof. For any nonzero $\chi \in \operatorname{Spec} A$ there is a $a_\chi \in A$ such that $\widehat{a}_\chi(\chi) \neq 0$ and so if we define $p_\chi = a_\chi a_\chi^*$ then it is positive-definite and $\widehat{p}_\chi = |\widehat{a}_\chi|^2$ is strictly nonzero in an open neighbourhood U_χ of χ. From this we get an open cover $\{U_\chi : \chi \in K\}$ of K and by compactness there are $\chi_1, \ldots, \chi_n \in K$ such that $K \subset \bigcup_{k=1}^n U_{\chi_k}$. Taking $p = p_{\chi_1} + \ldots + p_{\chi_n}$ finishes the proof. ■

Now we can for every $f \in C_c(\operatorname{Spec} A)$ find at least one positive-definite $p \in A$ such that $\widehat{p}$ is strictly nonzero on $\operatorname{supp}(f)$, so define a functional $\nu_\alpha : C_c(\operatorname{Spec} A) \to \mathbb{C}$ by

$$\nu_\alpha(f) = \mu_p(f/\widehat{p})$$

for any such $p \in A$. It is well-defined, for if $p, q \in A$ are two such positive-definite elements then $\operatorname{supp}(\widehat{p}\widehat{q}) = \operatorname{supp}(\widehat{p}) \cap \operatorname{supp}(\widehat{q}) \supset \operatorname{supp}(f)$, meaning that $f/(\widehat{p}\widehat{q}) \in C_c(\operatorname{Spec} A)$, so

$$\mu_p\Big(\frac{f}{\widehat{p}}\Big) = \mu_p\Big(\frac{f}{\widehat{p}\widehat{q}}\widehat{q}\Big) = \mu_q\Big(\frac{f}{\widehat{p}\widehat{q}}\widehat{p}\Big) = \mu_q\Big(\frac{f}{\widehat{q}}\Big).$$

It is also linear, for if $p \in A$ is positive-definite and strictly nonzero on $\operatorname{supp}(f_1) \cup \operatorname{supp}(f_2) \supset \operatorname{supp}(f_1 + f_2)$, $f_1, f_2 \in C_c(\operatorname{Spec} A)$, then the same holds on both $\operatorname{supp}(f_1)$ and $\operatorname{supp}(f_2)$, so

$$\mu_p\Big(\frac{\lambda_1 f_1 + \lambda_2 f_2}{\widehat{p}}\Big) = \lambda_1 \mu_p\Big(\frac{f_1}{\widehat{p}}\Big) + \lambda_2 \mu_p\Big(\frac{f_2}{\widehat{p}}\Big).$$

Lastly, it is continuous in the inductive topology on $C_c(\operatorname{Spec} A)$. Indeed, if $K \subset \operatorname{Spec} A$ is compact and f is supported on a compact subset of K then for a positive-definite $p \in A$ as before,

$$|\nu_\alpha(f)| = |\mu_p\Big(\frac{f}{\widehat{p}}\Big)| \le \frac{\|\mu_p\|}{\delta_p} \|f\|_\infty .$$

Here, $\delta_p = \inf_{\alpha \in A} \widehat{p}(\alpha) > 0$ and since $\nu_\alpha(f)$ is independent of p then this bound implies continuity. From the proof of lemma A.5 we can take p such that $\widehat{p}$ is positive, so we now have a positive functional $\nu_\alpha \in C_c(\operatorname{Spec} A)$. By the Riesz representation theorem, ν_α defines a unique positive Radon measure on $\operatorname{Spec} A$ and we write

$$\nu_\alpha(f) = \int_{\operatorname{Spec} A} f \, d\nu_\alpha \,, \quad f \in C_c(\operatorname{Spec} A).$$

We can extend this functional to the Gelfand transform of elements in the ideal $\mathfrak{p}_\alpha$ in A generated by the positive-definite elements $p \in A$: If $f \in C_b(\operatorname{Spec} A)$ then $\widehat{p}$ is strictly nonzero on the support of $f\widehat{p}$, so

$$\int_{\operatorname{Spec} A} f \widehat{p} \, d\nu_\alpha = \int_{\operatorname{Spec} A} f \, d\mu_p$$

and in particular,

$$\int_{\operatorname{Spec} A} \widehat{x}\widehat{p} \, d\nu_\alpha = \int_{\operatorname{Spec} A} \widehat{x} \, d\mu_p = \alpha(px)$$

for all $x \in A$.

B.2 Proof of the Full Plancherel-Godement Theorem

Recall the original setting, where A is a complex involutive algebra and $B_\alpha \subset B(\mathscr{H}_\alpha)$ is a C^*-algebra. We saw that a $*$-positive functional $\alpha \in A^*$ lifts to a functional $\beta \in B_\alpha^*$ and by the proof for C^*-algebras, there is a unique positive Radon measure ν_β on $\operatorname{Spec} B_\alpha$ such that

$$\beta(qb) = \int_{\operatorname{Spec} B_\alpha} \widehat{b}\widehat{q} \, d\nu_\beta$$

for every positive-definite $q \in B_\alpha$ and every $b \in B_\alpha$. To finish the proof of the theorem we translate the measure ν_β back to $\operatorname{Spec} A$. First we define the space Ω_α and prove that it is locally compact.

The $*$-representation $\pi : A \to B_\alpha$ dualizes to a map on spectra,

$$\pi^* : \operatorname{Spec} B_\alpha \longrightarrow \operatorname{Spec} A \,,$$

given by precomposing with π. If $\chi \in \operatorname{Spec} B_\alpha$ then χ is uniquely defined on the dense subspace

$\pi(A) \subset B_\alpha$, so it is uniquely defined by $\pi^*\chi = \chi \circ \pi \in \operatorname{Spec} A$, meaning that π^* is injective. Now define

$$\Omega_\alpha = \pi^*(\operatorname{Spec} B_\alpha).$$

The space $\operatorname{Spec} B_\alpha$ is locally compact by the Banach-Alaoglu theorem since B_α is a Banach space and the spectrum is a subspace of the closed unit ball in B_α^*. For Ω_α to be locally compact, it thus suffices to show that π^* is a homeomorphism.

Lemma B.6. *The map* $\pi^* : \operatorname{Spec} B_\alpha \to \operatorname{Spec} A$ *is a homeomorphism onto* $\Omega_\alpha \subset \operatorname{Spec} A$.

Proof. If $\chi_\lambda \to \chi$ in $\operatorname{Spec} B_\alpha$, then the same holds on the subspace $\pi(A) \subset B_\alpha$, which is equivalent to $\pi^*\chi_\lambda \to \pi^*\chi$ in $\operatorname{Spec} A$. Thus π^* is continuous. Conversly, suppose that $\pi^*\chi_\lambda \to \pi^*\chi$ in $\Omega_\alpha \subset \operatorname{Spec} A$. Then $\chi_\lambda \to \chi$ on the dense subspace $\pi(A) \subset B_\alpha$. If $b \in B_\alpha$ and a_τ is a net in A such that $\pi(a_\tau) \to b$, then

$$|\chi_\lambda(b) - \chi(b)| \le |\chi_\lambda(b) - \chi_\lambda(\pi(a_\tau))| + |\chi_\lambda(\pi(a_\tau)) - \chi(\pi(a_\tau))| + |\chi(\pi(a_\tau)) - \chi(b)|.$$

Choosing λ and τ such that each term on the RHS is less than $\varepsilon > 0$, then $|\chi_\lambda(b) - \chi(b)| < 3\varepsilon$ and so $\chi_\lambda(b) \to \chi(b)$. Since $b \in B_\alpha$ was arbitrary then $\chi_\lambda \to \chi$ in $\operatorname{Spec} B_\alpha$ and $(\pi^*)^{-1}$ is continuous. ∎

Taking the positive Radon measure ν_β on $\operatorname{Spec} B_\alpha$, we can then push it forward to a unique positive Radon measure $\nu_\alpha = (\pi^*)_*\nu_\beta$ on $\Omega_\alpha \subset \operatorname{Spec} A$. If $y \in \mathfrak{p}_A$ then $\widehat{y}(\pi^*\chi) = \pi^*\chi(y) = \chi(\pi(y)) = \widehat{\pi(y)}(\chi)$, so by the definition of the pushforward we have

$$\int_{\operatorname{Spec} A} \widehat{y}\, d\nu_\alpha = \int_{\operatorname{Spec} B_\alpha} \widehat{\pi(y)}\, d\nu_\beta .$$

Finally, if $p \in A$ is positive-definite with respect to α then $\pi(p) \in B_\alpha$ is positive-definite with respect to β by definition, so π restricts to a map $\pi : \mathfrak{p}_A \to \mathfrak{p}_{B_\alpha}$. This means in particular that

$$\alpha(px) = \beta(\pi(p)\pi(x)) = \int_{\operatorname{Spec} B_\alpha} \widehat{\pi(x)}\widehat{\pi(p)}\, d\nu_\beta = \int_{\Omega_\alpha} \widehat{x}\widehat{p}\, d\nu_\alpha ,$$

concluding the proof of the Plancherel-Godement theorem.

References

[1] Jean-Philippe Anker. "The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan". In: *Journal of Functional Analysis* 96.2 (1991), pp. 331–349. ISSN: 0022-1236. DOI: https://doi.org/10.1016/0022-1236(91)90065-D. URL: http://www.sciencedirect.com/science/article/pii/002212369190065D.

[2] Harry Bateman. *Tables of Integral Transforms [Volume II]*. 1954.

[3] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. "Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets". In: *Proceedings of the London Mathematical Society* 116.4 (Dec. 2017), pp. 957–996. ISSN: 0024-6115. DOI: 10.1112/plms.12091. URL: http://dx.doi.org/10.1112/plms.12091.

[4] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. *Aperiodic order and spherical diffraction, II: Translation bounded measures on homogeneous spaces*. 2017. arXiv: 1704.00302 [math.DS].

[5] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. *Aperiodic order and spherical diffraction, III: The shadow transform and the diffraction formula*. 2020. arXiv: 2002.05618 [math.DS].

[6] Nicole Bopp. *Les distibutions de type positif relativement á un group fini d'isométries*. Lecture Notes in Mathematics: Analyse Harmonique sur le Groupes de Lie II. Springer-Verlag, 1979.

[7] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. 1995.

[8] William Fulton. *Algebraic curves - an introduction to algebraic geometry*. 2008.

[9] Israel Gelfand and Georgiy Shilov. *Generalized Functions, Volume 2: Spaces of Fundamental and Generalized Functions*. 1968.

[10] Israel Gelfand and Naum Vilenkin. *Generalized Functions, Volume 4*. 1964.

[11] A. P. Guinand. "Concordance and the harmonic analysis of sequences". In: *Acta Math.* 101.3-4 (1959), pp. 235–271. DOI: 10.1007/BF02559556. URL: https://doi.org/10.1007/BF02559556.

[12] Sigurdur Helgason. *Integral Geometry and Radon Transforms*. Jan. 2011. DOI: 10.1007/978-1-4419-6055-9.

[13] Jay Jorgenson and Serge Lang. *Spherical Inversion on* $\mathrm{SL}_n(\mathbb{R})$. 2001.

[14] Yves F. Meyer. "Measures with locally finite support and spectrum". In: *Proceedings of the National Academy of Sciences* 113.12 (2016), pp. 3152–3158. ISSN: 0027-8424. DOI: 10.1073/pnas.1600685113. eprint: https://www.pnas.org/content/113/12/3152.full.pdf. URL: https://www.pnas.org/content/113/12/3152.

[15] Sanjoy Pusti. "An analogue of Krein's theorem for semisimple Lie groups". In: *Pacific Journal of Mathematics* 254 (Dec. 2011), pp. 381–395. DOI: 10.2140/pjm.2011.254.381.