THESIS FOR THE DOCTORAL DEGREE OF PHILOSOPHY

Combinatorics of solvable lattice models with a reflecting end

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Abstract

In this thesis, we study some exactly solvable, quantum integrable lattice models. Izergin proved a determinant formula for the partition function of the six-vertex (6V) model on an $n \times n$ lattice with the domain wall boundary conditions (DWBC) of Korepin. The method has become a useful tool to study the partition functions of similar models. The determinant formula has also proved useful for seemingly unrelated questions. In particular, by specializing the parameters in Izergin's determinant formula, Kuperberg was able to give a formula for the number of alternating sign matrices (ASMs).

Bazhanov and Mangazeev introduced special polynomials, including p_n and q_n, that can be used to express certain ground state eigenvector components for the supersymmetric XYZ spin chain of odd length. In Paper I, we find explicit combinatorial expressions for the polynomials q_n in terms of the three-color model with DWBC and a (diagonal) reflecting end. The connection emerges by specializing the parameters in the partition function of the eight-vertex solid-on-solid (8VSOS) model with DWBC and a (diagonal) reflecting end in Kuperberg's way. As a consequence, we find results for the three-color model, including the number of states with a given number of faces of each color. In Paper II, we perform a similar study of the polynomials p_n. To get the connection to the 8VSOS model, we specialize all parameters except one in Kuperberg's way.

By using the Izergin–Korepin method in Paper III, we find a determinant formula for the partition function of the trigonometric 6V model with DWBC and a partially (triangular) reflecting end on a $2n \times m$ lattice, $m \leq n$. Thereafter we use Kuperberg's specialization of the parameters to find an explicit expression for the number of states of the model as a determinant of Wilson polynomials. We relate this to a type of ASM-like matrices.

Keywords: six-vertex model, eight-vertex SOS model, three-color model, reflecting end, domain wall boundary conditions, partition function, determinant formula, XYZ spin chain, alternating sign matrices, special polynomials, positive coefficients.

List of papers

This doctoral thesis is based on the following papers:

Paper I. L. Hietala, *A combinatorial description of certain polynomials related to the XYZ spin chain*, SIGMA **16** (2020), 101, 26 pages, arXiv: 2004.09924v2, DOI: 10.3842/SIGMA.2020.101.

Paper II. L. Hietala, *A combinatorial description of certain polynomials related to the XYZ spin chain. II. The polynomials* p_n, preprint, arXiv: 2104.04651.

Paper III. L. Hietala, *Exact results for the six-vertex model with domain wall boundary conditions and a partially reflecting end*, preprint, arXiv: 2104.05389.

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Contents

1 Introduction

In statistical mechanics, the goal is to describe the macroscopic properties of a system by modeling the microscopic interaction between its components. Each possible state S of a model has a weight $W(S)$ assigned to it. The *partition function* of a model is the sum of the weights of all possible states, given by

$$Z = \sum_{\text{states}} W(S). \tag{1.1}$$

Then $W(S)/Z$ is the probability of finding the system in a certain state S.

Some models are *exactly solvable*, which means that one can find the thermodynamic behavior when the system size tends to infinity. In particular the free energy of the model is determined by the behaviour of the partition function in the thermodynamic limit. In this thesis we study even more special situations, where it is possible to find an exact expression for the partition function even for finite system sizes.

Solvability often depends on *quantum integrability*, which in situations of interest to us can be described by a family of commuting transfer matrices. One can also require a local condition, namely, a description in terms of an R-matrix obeying the Yang–Baxter equation, which guarantees the macroscopic property of commuting transfer matrices (see further Section 3.1).

We start in Section 2 by introducing some different lattice models and boundary conditions of interest for this thesis. The lattice models considered in this thesis are quantum integrable and exactly solved for finite system sizes. In Section 3, we describe the six-vertex model with domain wall boundary conditions more carefully, and look into its connections to alternating sign matrices and the XXZ spin chain. We also describe the six-vertex model with domain wall boundary conditions and a reflecting end. A variant of this model is studied in Paper III. Then, in Section 4, we describe the eight-vertex solid-on-solid model with domain wall boundary conditions and a reflecting end. This is the model of consideration in Paper I and II. Many ideas important in the analysis of the six-vertex model generalize to the eight-vertex solid-on-solid model as well. Furthermore we look at the connection between the eight-vertex solid-on-solid model and the three-color model. In Section 5, we discuss certain polynomials, showing up in different contexts. There are connections to the XYZ spin chain as well as to the eight-vertex model and the three-color model. Finally in Section 6, we summarize the results of the included papers, and in Section 7, we discuss ideas for future investigations.

2 Solvable lattice models

In this section, we introduce the lattice models that are relevant for this thesis. As we will see in later sections, these models are exactly solvable.

2.1 Vertex models

The vertex models that we consider in this thesis are models on a (piece of a) square lattice (i.e. a lattice with $m \times n$ vertices), with edges connecting nearest neighbours. A state of a model is a lattice with a spin ± 1 assigned to each edge. Graphically a state can be represented by giving each line a positive direction, which we indicate by an arrow at the end of the line (this will be useful later), and then spin $+1$ corresponds to an arrow halfway the edge pointing in the positive direction of the line, and spin -1 corresponds to an arrow pointing in the negative direction. In the easiest examples of the vertex models that we consider, we choose the positive directions to be up and to the right, i.e. positive spin corresponds to arrows pointing upwards or to the right, and negative spin corresponds to arrows pointing downwards or to the left. Each vertex has a local weight $w\begin{pmatrix} & \beta' & \\ \alpha & & \alpha' \\ & \beta & \end{pmatrix}$ that depends on the spins $\alpha, \beta, \alpha', \beta'$ on the surrounding edges as in Figure 2.1. The weight of a state is the product of all local weights.

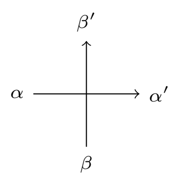


Figure 2.1: A vertex with spins $\alpha, \beta, \alpha', \beta' = \pm 1$ on the surrounding edges.

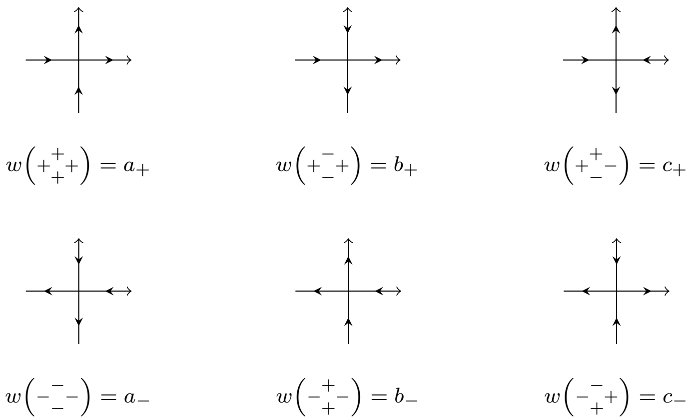


Figure 2.2: The possible vertices and their vertex weights for the 6V model.

One of the first examples of a *six-vertex* (6V) *model* was introduced to model hydrogen bonds in ice. The ice model and some other special cases of the 6V model with periodic boundary conditions in both directions was solved in the thermodynamic limit by Lieb [25] in 1967, by using the Bethe ansatz (see e.g. [1, 24]). Later the same year, Sutherland [40] solved the general 6V model with periodic boundary conditions.

A square lattice can be used for a two-dimensional approximation of the ice structure. The vertices of the lattice then represent oxygen atoms. Each oxygen atom has exactly two hydrogen atoms close by, sitting on two of the neighbouring edges. Together they form a water molecule. Furthermore, each oxygen atom has bonds to two other hydrogen atoms belonging to two other water molecules. Hence on each edge there is exactly one hydrogen atom. The spins describe where the hydrogen atoms are, with an arrow pointing inwards to the vertex if the hydrogen atom is "closest to" that oxygen atom. Thus each vertex has exactly two arrows pointing inwards and two arrows pointing outwards. This imposes the ice rule: at each vertex with spins α, β, α' and β' as in Figure 2.1, the equation

$$\alpha + \beta = \alpha' + \beta' \tag{2.1}$$

must hold. This yields six possible types of vertices, namely, the vertices in Figure 2.2, with nonzero local weights $a_\pm$, $b_\pm$ and $c_\pm$. In the original *ice model*, all vertices, and hence all states, have the same weight. Different choices of the weights yield models for ferroelectric and antiferroelectric materials.

A generalization of the 6V model is the *eight-vertex* (8V) *model*, where the ice rule only holds modulo 2. This allows for two additional possible configurations

Figure 2.3: The two additional vertices with their vertex weights for the 8V model.

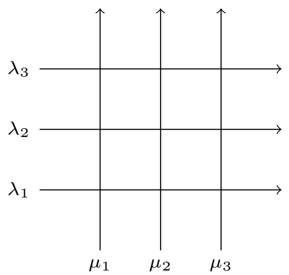


Figure 2.4: The inhomogeneous 6V model with spectral parameters λ_i and μ_j.

around the vertices, namely, a sink and a source, where either all arrows are pointing inwards or all arrows are pointing outwards from the vertex, see Figure 2.3. These two vertex weights are called $d_\pm$.

The case $a_+ = a_-$, $b_+ = b_-$ and $c_+ = c_-$ is called the *symmetric* 6V model. As we will see in Section 3.1, the 6V model is quantum integrable. To explain this fact and to study the model, the *inhomogeneous* 6V model is often a useful tool. In this generalization of the 6V model, we assign a spectral parameter λ_i to each horizontal line of the lattice, and a spectral parameter μ_j to each vertical line, see Figure 2.4. The weight of a vertex depends on the spectral parameters λ_i and μ_j on the lines going through the vertex, as in Figure 2.5. The vertex weights are then given by trigonometric functions $a_\pm(\lambda_i - \mu_j)$, $b_\pm(\lambda_i - \mu_j)$, $c_\pm(\lambda_i - \mu_j)$, see (3.4). A special case is when the weights are taken to be rational functions.

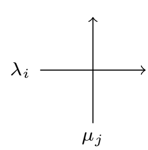


Figure 2.5: A vertex with spectral parameters λ_i and μ_j, and weight $w(\lambda_i - \mu_j)$.

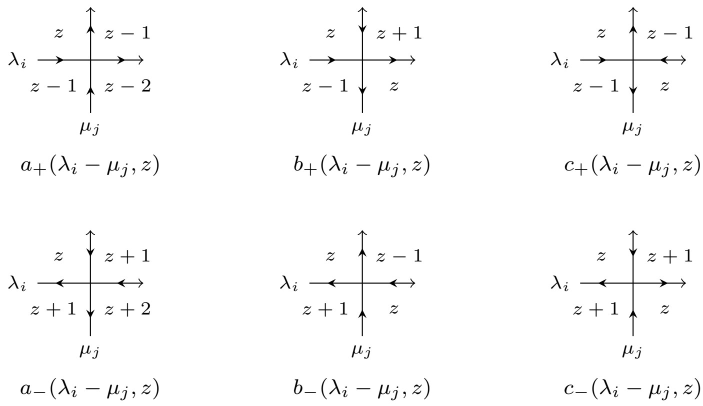


Figure 2.6: The possible vertices and their vertex weights for the 6VSOS model.

In the *six-vertex solid-on-solid* (6VSOS) *model*, also called the *trigonometric solid-on-solid model*, a height is assigned to each face, in addition to the spectral parameters on the lines. The heights z take values in $\rho + \mathbb{Z}$, where $\rho \in \mathbb{C}$ is a reference height called the dynamical parameter. For $z = \rho + a$, we sometimes also refer to a as the height. Going around a vertex clockwise, the height decreases by 1 when crossing an arrow pointing outwards and increases by 1 when crossing an arrow pointing inwards. The ice rule ensures that this description of the heights is well-defined. Going around a vertex, we will always come back to the same height. Therefore in a state of a lattice, the heights are determined by the height in one place, for instance we can take the height in the upper left corner to be ρ. Each vertex weight depends on the height z in one of the adjacent faces, which we take to be the face in the upper left corner. The vertex weights $a_\pm(\lambda_i - \mu_j, z)$, $b_\pm(\lambda_i - \mu_j, z)$ and $c_\pm(\lambda_i - \mu_j, z)$ are trigonometric functions of λ_i, μ_j and z, see Figure 2.6.

The same construction is not possible for the 8V model. The sinks and sources only enable the heights to be well-defined modulo 4. Nevertheless there is an SOS model related to the 8V model, namely, the *eight-vertex solid-on-solid* (8VSOS) *model*, which was introduced by Baxter [4] to solve the 8V model. The name is a bit misleading, since it has only six different local states. Therefore the model is also called the *elliptic solid-on-solid model*. The model is a two parameter generalization of the 6V model, the states are the same as in the 6VSOS model, but the weights $a_\pm(\lambda_i - \mu_j, z)$, $b_\pm(\lambda_i - \mu_j, z)$, $c_\pm(\lambda_i - \mu_j, z)$ are elliptic functions, see (4.10).

2.2 Boundary conditions

By fixing the spins on the external edges, we impose boundary conditions on our lattice models. Because of the ice rule, the total number of edges with spin $+1$ on the left and bottom boundaries must equal the total number of edges with spin $+1$ on the top and right boundaries. Likewise the number of edges with spin -1 on the left and the bottom must equal the number of edges with spin -1 on the top and the right.

One option is to take periodic boundary conditions in both directions (i.e. wrapping the lattice on a torus). This case was studied by Lieb and Sutherland for the 6V model, and by Baxter for the 8V model (see e.g. [25, 40, 3]). In this thesis, we focus on fixed boundary conditions. One important case of fixed boundary conditions are the *domain wall boundary conditions* (DWBC) [21], where all edges on the left and on the top have spin -1, and all edges on the right and at the bottom have spin $+1$, i.e. ingoing arrows on the top and the bottom and outgoing arrows to the left and the right, as in Figure 2.7. The ice rule forces a lattice with DWBC to be an $n \times n$ lattice.

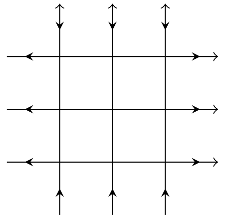


Figure 2.7: Lattice with DWBC in the case $n = 3$.

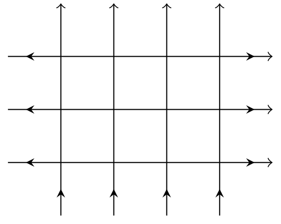


Figure 2.8: Lattice with partial DWBC. The spins on the top are dictated by the ice rule.

In the case of an $m \times n$ lattice where $m \neq n$, DWBC are impossible. In this case we can have *partial DWBC* [16], which are normal DWBC on three of the sides and on one side we leave the spins free, see Figure 2.8. In each configuration, the spins on the free boundary must add up such that the ice rule can be followed at all vertices in the lattice.

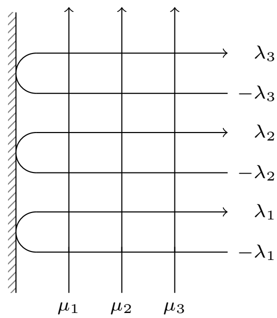


Figure 2.9: A lattice with one reflecting end, and spectral parameters λ_i and μ_j, for $n = 3$.

Consider a $2n \times m$ lattice where the horizontal lines are connected pairwise at the left boundary, as in Figure 2.9. Such a lattice is said to have a *reflecting end*. Each pair of horizontal lines can be thought of as one single line turning at a wall, i.e. each double line first has the positive direction to the left on the lower part of the line, then turns and has the positive direction to the right on the upper part of the line. The spectral parameters at the horizontal lines are $-\lambda_i$ on the lower part of a double line and λ_i on the upper part.

Each edge at the turn can have either spin $+1$ or -1, which in the case of diagonal reflection (see Section 3.2) gives rise to two types of boundary weights, $k_\pm$, see Figure 2.10. Here the spin is conserved through the turn, meaning that one spin arrow points inwards towards the lattice and the other arrow points outwards. This is the case that we study in Paper I and II. It is also possible to allow for turns that can absorb or create extra spin arrows, i.e. where both arrows point in the same direction, see Figure 2.10. This means that the spin changes in the turn. In the case where we do not allow for absorption of spin arrows, but only for the other three types of turns, we get a triangular reflection matrix (see (3.15)). This choice is considered in Paper III.

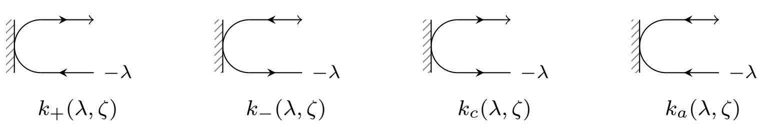


Figure 2.10: The possible boundary configurations and their boundary weights in the 6V model with a reflecting end.

For the 8VSOS model with a diagonal reflection matrix, the boundary weights depend on the height parameter outside the turn, which is the same for all turns, i.e. if the height ρ is fixed in the upper left corner of the lattice, then all boundary weights depend on this parameter. In the general (non-diagonal) case, the heights outside the turns along the reflecting boundary differ[1]. For the 8VSOS model, we only consider diagonal reflection in this thesis. The boundary weights also depend on the spectral parameter λ_i on the line, and a fixed boundary parameter $\zeta \in \mathbb{C}$, which we can think of as sitting on the reflective wall, see Figure 2.11. In the 8VSOS model, the weights $k_\pm(\lambda_i, \rho, \zeta)$ are elliptic functions, see (4.10). The 6V model with trigonometric weights is recovered in the limit $\rho \to \infty$ (some more technicalities are needed, see Section 4).

Figure 2.11: The possible boundary configurations and their boundary weights for the reflecting end in the 8VSOS model with a diagonal reflection matrix.

(a) $w(\lambda_i - \mu_j, z)$ (b) $w(\lambda_i + \mu_j, z)$

Figure 2.12: The different vertices depending on the orientation of the row in the 8VSOS model with a reflecting end, with spectral parameters λ_i and μ_j and height z.

The local vertex weights should always be read off with the positive directions up and to the right. In the case of a reflecting end, we need to differentiate between the vertices on the horizontal lines directed to the left and on those directed to the right. The vertices in the upper part of a double row are those depicted in Figure 2.6, and the vertices in the lower part are the same, but tilted 90 degrees counterclockwise, as in Figure 2.12b. The weight of the vertex in the upper part of a double row (see Figure 2.12a) is $w(\lambda_i - \mu_j, z)$, and for the vertex in the lower part of a double row (see Figure 2.12b), the weight is $w(\mu_j - (-\lambda_i), z) = w(\lambda_i + \mu_j, z)$, where w is one of $a_\pm, b_\pm$ or $c_\pm$.

[1] In the general case, it seems better to define the weights to depend on the heights inside the turns, to get commuting transfer matrices (cf. (3.13)) and to be able to find solutions for the reflection equation (4.7), (see further [39]). For diagonal reflection it is equivalent to let the weights depend on the heights outside the turn.

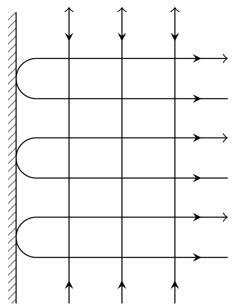


Figure 2.13: A lattice with DWBC and a reflecting end in the case $n = 3$.

In Paper I and II, we consider the 8VSOS model with one diagonal reflecting end and DWBC on the other three boundaries. Pictorially, in terms of arrows, the DWBC are the same as in the case without the reflecting end, the arrows point inwards on the top and bottom edges and outwards on the right boundary, see Figure 2.13. In terms of spins on the horizontal double lines with alternating orientation, this means that on the right boundary, we have spin -1 on the lower part of every double line (where the positive direction is to the left), and spin $+1$ on the upper part (with positive direction to the right). On the bottom, the edges have spin $+1$, and on the top, the edges have spin -1. The ice rule forces such a lattice to have n vertical lines and $2n$ horizontal lines.

In Paper III, we consider the 6V model on a $2n \times m$ lattice, $m \leq n$, with one triangular reflecting end and DWBC on the other three boundaries. On the three boundaries with DWBC, the arrows are as in the $2n \times n$ case. The ice rule forces spin conservation at the vertices, so for a lattice of size $2n \times m$, $m \leq n$, with DWBC on three sides, we must allow for creation of arrows at the reflecting end, i.e. two arrows pointing inwards towards the lattice. We also allow for turns where the spins are conserved through the turn, such that one arrow goes inwards and one arrow goes outwards. A lattice with this type of boundary conditions is said to have DWBC and a partially reflecting end [17].

2.3 The three-color model

Another model on a square lattice is the *three-color model*, where the faces are filled with three different colors, which we call color 0, 1, and 2, such that adjacent faces have different colors. A weight t_i is then assigned to each face with color i. A state of the three-color model is called a three-coloring. Again, the weight of a state is the product of the local weights. If we reduce the heights

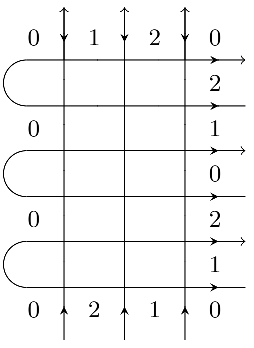


Figure 2.14: The DWBC and reflecting end for the three-color model of size $n = 3$, with colors 0, 1 and 2. The arrows on the edges show the corresponding boundary conditions in the 8VSOS model.

$z = \rho + a$ of the faces in the 8VSOS model to a modulo 3, the states of the 8VSOS model can be identified with states of the three-color model, for which the upper left corner has color 0 fixed. This bijection was found by Lenard [25, (note added in the proof)], and the three-color model where each color is given a weight was then introduced by Baxter [2]. The partition function of the three-color model, with the color in the upper left corner fixed, is

$$Z_n^{3C}(t_0, t_1, t_2) = \sum_{\text{states}} \prod_{\text{faces}} t_i. \tag{2.2}$$

In Paper I and II, we consider the three-color model on a square lattice with $(2n+1) \times (n+1)$ faces, corresponding to a limit of the 8VSOS model with DWBC and a reflecting end. For the three-color model, these boundary conditions correspond to the following rules for the colors (see Figure 2.14). In the upper left corner, we fix color 0. On three of the boundaries, the colors alternate cyclically. Starting from the upper left corner, going to the right, the colors increase in the order $0 < 1 < 2 < 0$, to reach $n \bmod 3$ in the upper right corner. From there, going down, the colors decrease down to $(-n) \bmod 3$ in the lower right corner. Continuing to the left, the colors increase again, up to 0 in the lower left corner. On the left side, every second face has color 0. Inside the turns the colors differ. A negative turn in the corresponding state of the 8VSOS model corresponds to color 1, and a positive turn corresponds to color 2.

In Paper II, we are restricted to special cases of the three-color model, where we also fix the colors along the second to last column on the right side. Starting from the bottom of that column, the colors 0, 1 and 2 change in ascending order modulo 3, except when it crosses the lth edge, where it decreases by $1 \bmod 3$ to then continue in ascending order.

3 The six-vertex model

In this section, we focus on the six-vertex (6V) model and its connections to alternating sign matrices and the XXZ spin chain. We start in the first two sections by giving an algebraic description of the 6V model with different types of boundary conditions important for this thesis. Thereafter we go through the proof of Izergin–Korepin's determinant formula for the partition function in Section 3.3. In Section 3.4 and Section 3.5, we introduce alternating sign matrices and discuss their connections to the 6V model, and in Section 3.6, we turn the attention to the XXZ spin chain.

3.1 Algebraic description of the six-vertex model

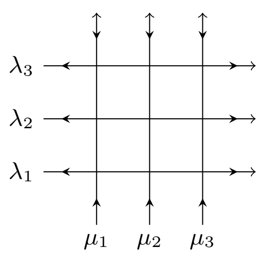


Figure 3.1: The 6V model with DWBC and spectral parameters λ_i and μ_j, for $n = 3$.

Let $q^x = e^{2\pi i\eta x}$, where $\eta \notin \mathbb{Z}$ is a fixed parameter. Throughout this section, let

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}. \tag{3.1}$$

Consider the (inhomogeneous) 6V model on an $n \times n$ lattice, as in Figure 3.1. To each line in the lattice, assign a two-dimensional complex vector space V

with basis vectors e_+ and e_-. The space $V \otimes V$ then has the four basis vectors $e_+ \otimes e_+$, $e_+ \otimes e_-$, $e_- \otimes e_+$ and $e_- \otimes e_-$. Given a parameter $\lambda \in \mathbb{C}$, define operators $R(\lambda) \in \mathrm{End}(V \otimes V)$ by

$$R(\lambda)(e_\alpha \otimes e_\beta) = \sum_{\alpha+\beta=\alpha'+\beta'} w\left(\alpha \,{}^{\beta'}_{\beta}\, \alpha'\right)(\lambda) e_{\alpha'} \otimes e_{\beta'}, \tag{3.2}$$

where $w\left(\alpha \,{}^{\beta'}_{\beta}\, \alpha'\right)(\lambda)$ is the weight $a_\pm(\lambda)$, $b_\pm(\lambda)$ or $c_\pm(\lambda)$ corresponding to a vertex with spins $\alpha, \beta, \alpha', \beta'$ on the surrounding edges as in Figure 2.2. The operator is called the R-matrix and is given by

$$R(\lambda) = \begin{pmatrix} a_+(\lambda) & 0 & 0 & 0 \\ 0 & b_+(\lambda) & c_-(\lambda) & 0 \\ 0 & c_+(\lambda) & b_-(\lambda) & 0 \\ 0 & 0 & 0 & a_-(\lambda) \end{pmatrix}, \tag{3.3}$$

where we take the weights to be parametrized as

$$\begin{aligned} &a_+(\lambda) = a_-(\lambda) = [\lambda + 1], \\ &b_+(\lambda) = b_-(\lambda) = [\lambda], \\ &c_+(\lambda) = q^{\lambda/2}, \qquad c_-(\lambda) = q^{-\lambda/2}. \end{aligned} \tag{3.4}$$

Remark 3.1. For DWBC, the weights above are equivalent to the symmetric case where $a_+(\lambda) = a_-(\lambda)$, $b_+(\lambda) = b_-(\lambda)$, $c_+(\lambda) = c_-(\lambda)$. This follows since the difference between the number of c_- vertices and c_+ vertices is equal for all states, see Section 3.4.

With this parametrization of the weights, the R-matrix satisfies the *Yang–Baxter equation* (YBE) on $V_1 \otimes V_2 \otimes V_3$ (where V_i are copies of V), i.e.

$$\begin{aligned} R_{12}(\lambda_1 - \lambda_2)&R_{13}(\lambda_1 - \lambda_3)R_{23}(\lambda_2 - \lambda_3) \\ &= R_{23}(\lambda_2 - \lambda_3)R_{13}(\lambda_1 - \lambda_3)R_{12}(\lambda_1 - \lambda_2), \end{aligned} \tag{3.5}$$

where the indices indicate on which spaces the R-matrix acts, i.e.

$$R_{12}(\lambda_1 - \lambda_2) = R(\lambda_1 - \lambda_2) \otimes \mathrm{Id}, \tag{3.6}$$

$$R_{23}(\lambda_2 - \lambda_3) = \mathrm{Id} \otimes R(\lambda_2 - \lambda_3). \tag{3.7}$$

In the same way, R_{13} acts on the first and third space. To make the definition precise, we first define a permutation operator $P \in \mathrm{End}(V \otimes V)$, by

$$P(e_\alpha \otimes e_\beta) = e_\beta \otimes e_\alpha. \tag{3.8}$$

Then

$$R_{13} = (\mathrm{Id} \otimes P)(R \otimes \mathrm{Id})(\mathrm{Id} \otimes P) \\ = (P \otimes \mathrm{Id})(\mathrm{Id} \otimes R)(P \otimes \mathrm{Id}). \tag{3.9}$$

The YBE is depicted in Figure 3.2.

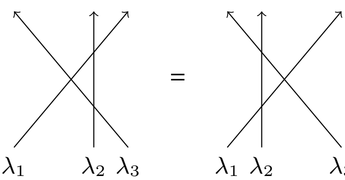


Figure 3.2: The Yang–Baxter equation.

The partition function of the 6V model with DWBC on an $n \times n$ lattice depends on the spectral parameters λ_i and μ_j, and is given by

$$Z_n(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) = \sum_{\text{states}} \prod_{\text{vertices}} w(\text{vertex}), \tag{3.10}$$

where $w(\text{vertex})$ is one of the local weights $a_\pm(\lambda_i - \mu_j)$, $b_\pm(\lambda_i - \mu_j)$ or $c_\pm(\lambda_i - \mu_j)$. It also depends implicitly on the parameter η.

Now consider one row of the lattice. Label the horizontal line by 0, and the vertical lines by $1, \ldots, n$, as in Figure 3.3. Then we define the *monodromy matrix* $T_0 \in \mathrm{End}(V_0 \otimes \cdots \otimes V_n)$ as

$$T_0(\lambda, \mu_1, \ldots, \mu_n) = R_{0n}(\lambda - \mu_n) \cdots R_{01}(\lambda - \mu_1), \tag{3.11}$$

where $R_{0i}(\lambda - \mu_i)$ is acting on the 0th and ith space, defined in a similar way as R_{13} in (3.9) above.

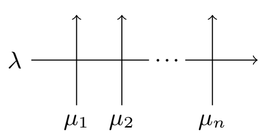


Figure 3.3: The monodromy matrix.

Writing the monodromy matrix as a matrix in the horizontal space V_0, we have

$$T_0(\lambda, \vec{\mu}) = \begin{pmatrix} A(\lambda, \vec{\mu}) & B(\lambda, \vec{\mu}) \\ C(\lambda, \vec{\mu}) & D(\lambda, \vec{\mu}) \end{pmatrix}_0 \tag{3.12}$$

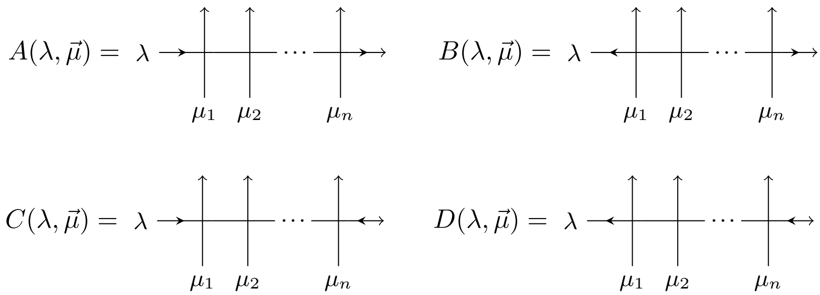


Figure 3.4: The entries of the monodromy matrix in the horizontal space.

where $A(\lambda, \vec{\mu})$, $B(\lambda, \vec{\mu})$, $C(\lambda, \vec{\mu})$ and $D(\lambda, \vec{\mu})$ are matrices corresponding to fixed spins on the left and right horizontal edges as in Figure 3.4.

Consider the $6V$ model with periodic boundary conditions on the horizontal lines, i.e. the left and right horizontal edges have the same spin. Then only $A(\lambda, \vec{\mu})$ and $D(\lambda, \vec{\mu})$ from the corresponding monodromy matrix (3.12) occur in the partition function. Now introduce the *transfer matrix* as the trace of the monodromy matrix in the horizontal space,

$$t(\lambda, \vec{\mu}) = \mathrm{tr}_0 T_0(\lambda, \vec{\mu}) = A(\lambda, \vec{\mu}) + D(\lambda, \vec{\mu}). \tag{3.13}$$

Transfer matrices for any two rows commute, i.e. $[t(\lambda, \vec{\mu}), t(\lambda', \vec{\mu})] = 0$. This follows from the YBE in a similar way as in the proof of Lemma 3.2 below. The commutativity of transfer matrices provides a mathematical formulation of quantum integrability for the 6V model. It is the main reason that the model is exactly solvable.

3.2 Algebraic description of the six-vertex model with a reflecting end

Consider the 6V model on a $2n \times m$ lattice where the horizontal lines are connected pairwise at the left boundary to form a double line, as in Figure 3.5. To each double line of the lattice we associate a copy of the two-dimensional vector space V, and define operators $R(\lambda) \in \mathrm{End}(V \otimes V)$ as in the previous section.

To describe the reflecting boundary, fix a parameter $\zeta \in \mathbb{C}$ and then define a diagonal operator $K(\lambda, \zeta) \in \mathrm{End}(V)$, such that it satisfies the *reflection equation*

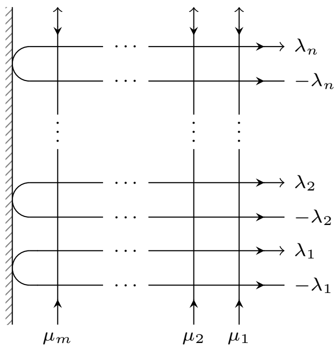


Figure 3.5: The 6V model with DWBC and a reflecting end. The parameters μ_i and λ_i are the spectral parameters.

for the R-matrix (3.3) on $V_0 \otimes V_{0'}$ [38], i.e.

$$R_{00'}(\lambda-\lambda')K_0(\lambda,\zeta)R_{0'0}(\lambda+\lambda')K_{0'}(\lambda',\zeta) \\ = K_{0'}(\lambda',\zeta)R_{00'}(\lambda+\lambda')K_0(\lambda,\zeta)R_{0'0}(\lambda-\lambda'), \quad (3.14)$$

where $K_0(\lambda,\zeta) = K(\lambda,\zeta)\otimes \mathrm{Id}$ and $K_{0'}(\lambda,\zeta) = \mathrm{Id}\otimes K(\lambda,\zeta)$, see Figure 3.6. The operator is called the K-matrix.

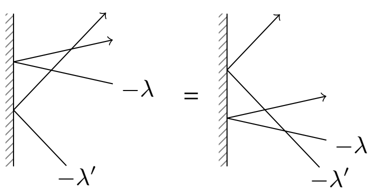


Figure 3.6: The reflection equation for the matrix $K(\lambda,\zeta)$.

The 6V model with a reflecting end is given by the R-matrix (3.3) which can be parametrized by the weights (3.4), and the K-matrix

$$K(\lambda,\zeta) = \begin{pmatrix} k_+(\lambda,\zeta) & k_c(\lambda,\zeta) \\ k_a(\lambda,\zeta) & k_-(\lambda,\zeta) \end{pmatrix}, \quad (3.15)$$

where the entries can be parametrized as

$$k_+(\lambda,\zeta) = q^{(\zeta-\lambda)/2}\,[\zeta+\lambda]\,, \qquad k_-(\lambda,\zeta) = q^{(\zeta+\lambda)/2}\,[\zeta-\lambda]\,, \quad (3.16)$$

$$k_c(\lambda,\zeta) = \varphi_c\,[2\lambda]\,, \qquad k_a(\lambda,\zeta) = \varphi_a\,[2\lambda]\,, \quad (3.17)$$

for fixed boundary parameters ζ, φ_c and φ_a. This yields a solution to the YBE

(3.5) and the reflection equation (3.14). The weights correspond to the states as in Figure 2.2 and in Figure 2.10. As seen in Figure 2.10, k_c can be thought of as a turn with creation of arrows and k_a can be thought of as a turn with absorption of arrows, hence the subscripts. For $\varphi_a = \varphi_c = 0$, we get a diagonal reflection matrix. In Paper III, we consider the 6V model with a reflecting end with an upper triangular K-matrix, i.e. where $\varphi_a = 0$, on a lattice of size $2n \times m$, $m \leq n$ (with the weights parametrized slightly differently).

The partition function of the 6V model with DWBC and a reflecting end depends on the vertex weights as well as on the boundary weights of the turns. Let $w(\text{vertex})$ be the local weight of a vertex, which depends on the λ_i and μ_j belonging to the lines passing through the vertex, and let $w(\text{turn})$ be the local weight of a turn, which depends on the λ_i belonging to the line through the turn and the boundary parameter ζ. Then the partition function is

$$Z_n(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m, \zeta) = \sum_{\text{states}} \prod_{\text{vertices}} w(\text{vertex}) \prod_{\text{turns}} w(\text{turn}). \qquad (3.18)$$

Implicitly the partition function also depends on η, φ_a and φ_c. The monodromy matrix and transfer matrix in the case of a reflecting end is discussed in Section 4.1.

3.3 The Izergin–Korepin determinant formula

By establishing properties which together determine the partition function, and then suggesting a formula which satisfies the conditions, Izergin and Korepin found a determinant formula for the partition function of the 6V model with DWBC. Korepin [21] found recurrence relations for the partition function, which Izergin [19, 20] was able to solve in terms of a determinant formula. We will now sketch the proof.

Lemma 3.2. *The partition function $Z_n(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ of the 6V model with DWBC is symmetric in the λ_i's and the μ_j's respectively.*

Proof. Consider two adjacent horizontal rows with spectral parameters λ and λ'. Insert an extra vertex to the left of the lattice, as in Figure 3.7. Since we have DWBC, this will be a vertex with weight $a_+(\lambda - \lambda')$. By the YBE, the extra vertex can be moved through the whole lattice and end up to the right, where it can be removed. On the right side, the extra vertex has the weight $a_-(\lambda - \lambda')$. Since $a_+(\lambda) = a_-(\lambda) \neq 0$, this procedure switches the λ and the λ'. The same argument applies to the μ_j's. □

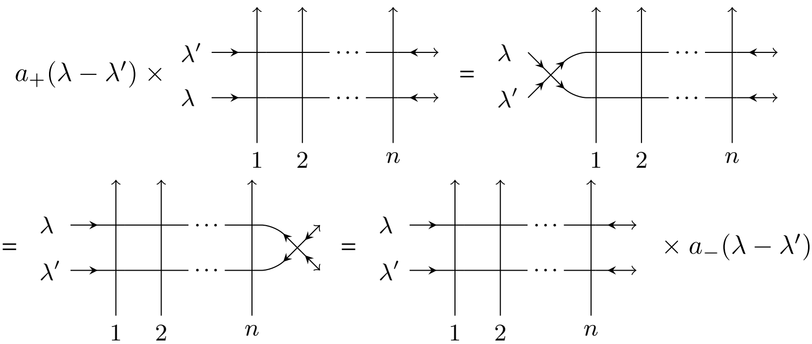


Figure 3.7: In case of DWBC, an extra vertex can be moved through the lattice by using the YBE.

The argument in the proof above is sometimes called the train argument.

Lemma 3.3 (Korepin). *Specifying $\lambda_i = \mu_j$, or $\lambda_i = \mu_j - 1$, yields a recurrence relation for the partition function.*

Proof. First specify $\lambda_1 = \mu_1$. Then for any state with nonzero weight, the vertex in the lower left corner must be a c_- vertex, because of the ice rule. This happens because $a_\pm$ vertices are not possible in that corner, and terms including $b_\pm(0)$ disappear. With this specification, the rest of the bottom row must have a_+ vertices and the left column must have a_- vertices. In this way, a part of the lattice is frozen. The remaining lattice is now an $(n-1) \times (n-1)$ lattice with DWBC. Hence

$$Z_n(\vec{\lambda}, \vec{\mu}) = c_-(0) \prod_{k=2}^{n} a_-(\lambda_k - \mu_1) a_+(\lambda_1 - \mu_k) \\ \times Z_{n-1}(\lambda_2, \dots, \lambda_n, \mu_2, \dots, \mu_n). \tag{3.19}$$

Similarly if $\lambda_n = \mu_1 - 1$, then

$$Z_n(\vec{\lambda}, \vec{\mu}) = c_-(-1) \prod_{k=1}^{n-1} b_-(\lambda_k - \mu_1) \prod_{k=2}^{n} b_+(\lambda_n - \mu_k) \\ \times Z_{n-1}(\lambda_1, \dots, \lambda_{n-1}, \mu_2, \dots, \mu_n). \tag{3.20}$$

The general case follows by the symmetries in Lemma 3.2. □

Lemma 3.4. *The normalized partition function $q^{n\lambda_n/2} Z_n(\vec{\lambda}, \vec{\mu})$ is a polynomial in q^{λ_n} of degree at most $n-1$.*

Proof. The only row containing q^{λ_n} is the upper row. Multiplying all weights in the upper row by $q^{\lambda_n/2}$, these new weights are linear in q^{λ_n} (if q^{λ_n} appears in the weight at all). Hence the normalized partition function is a polynomial in q^{λ_n}. In the upper row, there is exactly one c_- vertex, and the rest of the vertices in the upper row are a_- or b_+ vertices. Multiplying the weight of the c_- vertex with $q^{\lambda_n/2}$, yields a constant (seen as a polynomial in q^{λ_n}), so the polynomial has degree at most $n-1$. □

The partition function for a system of size $n = 1$ has only one term. By Lagrange interpolation it is possible to determine $Z_n(\vec{\lambda}, \vec{\mu})$ inductively. Because of the degree of the normalized partition function, one needs n points for the Lagrange interpolation. The recurrence relations of Lemma 3.3 are enough to determine the general partition function. Izergin [19, 20] solved this recurrence relation.

Theorem 3.5 (Izergin, Korepin)**.** *The partition function of the 6V model with DWBC is given by*

$$\begin{aligned} &Z_n(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \\ &\qquad = \frac{\prod_{i=1}^n q^{(\mu_i-\lambda_i)/2} \prod_{i,j=1}^n ([\lambda_i - \mu_j][\lambda_i - \mu_j + 1])}{\prod_{1\le i,j\le n}([\lambda_i - \lambda_j][\mu_j - \mu_i])} \det_{1\le i,j\le n} K, \end{aligned} \tag{3.21}$$

where

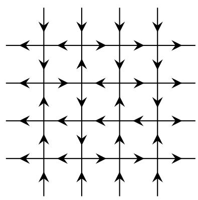
$$K_{ij} = \frac{1}{[\lambda_i - \mu_j][\lambda_i - \mu_j + 1]}. \tag{3.22}$$

With the same method, Tsuchiya [41] was able to find a determinant formula for the partition function of the 6V model with DWBC and a reflecting end. It was generalized to the 8VSOS model by Filali [15], see Theorem 4.6. In Paper III, we use the Izergin–Korepin method to find a determinant formula for the 6V model with DWBC and a partially reflecting end for a lattice of size $2n \times m$, for $m \le n$.

3.4 Alternating sign matrices

An *alternating sign matrix* (ASM) is a matrix with entries $-1, 0, 1$, such that in each row and each column, the nonzero entries have alternating signs, and the sum of the entries in each row and each column is 1. This means that the first and last nonzero entry of each row and each column is 1. In particular, the first and last row and column have exactly one nonzero entry each.

Kuperberg [22] studied the bijection between ASMs and the states of the 6V model with DWBC. In the 6V model, focus on the vertices with weights $c_\pm$. The c_+ and c_- vertices must alternate, and for DWBC, each row and each column must have one more c_- vertex than c_+ vertices. Now consider the mapping from a state of the $n \times n$ lattice with DWBC to an $n \times n$ matrix, where each entry of the matrix corresponds to a vertex of the lattice in such a way that a c_- vertex is mapped to $+1$, a c_+ vertex is mapped to -1, and all other vertices are mapped to 0, see Figure 3.8. This mapping is a bijection from the states of the 6V model with DWBC to the ASMs, the ice rule determines the inverse uniquely.



$$\longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 3.8: The bijection between a state of the 6V model with DWBC and an ASM.

Mills, Robbins and Rumsey [27] conjectured an expression for the number of ASMs.

Theorem 3.6 (Zeilberger)**.** *The number of $n \times n$ ASMs is*

$$A_n = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{3.23}$$

Zeilberger [42] proved the ASM conjecture, and Kuperberg [22] gave another, much simpler proof, using the 6V model.

In any $n \times n$ state of the 6V model with DWBC, the number of a_+ vertices and a_- vertices are equal, the number of b_+ vertices and b_- vertices are equal as well, and there are always n more c_- vertices than c_+ vertices, see [9, Section 7.1]. By specializing $\eta = -2/3$, $\lambda_i = -1/2$ and $\mu_j = 0$, Kuperberg was able to count the number of ASMs. In this case, the partition function only depends on $(c_-(1/2))^n = q^{n/4}$. All other factors appear in pairs which cancel each other. Hence $q^{-n/4} Z_n(-1/2, \ldots, -1/2, 0, \ldots, 0)$ counts the number of states. Unfortunately, this specification of the λ_i's and μ_j's makes the Izergin–Korepin determinant of Theorem 3.5 singular. Therefore Kuperberg instead considered the partition function for $\lambda_i = -1/2 + i\epsilon$ and $\mu_j = (1-j)\epsilon$. In this situation he could compute the determinant explicitly. In the limit $\epsilon \to 0$, the formula in Theorem 3.6 follows.

3.5 U-turn alternating sign matrices

In addition to the usual ASMs, Kuperberg [23] investigated several other types of ASMs connected to different types of boundary conditions, among them, the *U-turn alternating sign matrices* (UASMs) which generalize the *vertically symmetric alternating sign matrices* (VSASMs).

A UASM is a $2n \times n$ ASM with a U-turn on the left side. More precisely, consider matrices consisting of elements 0, -1 and 1, such that the nonzero elements alternate in signs vertically and horizontally. The sum of the elements of each column is 1. Horizontally, connect the rows pairwise on the left side to form a double row, see Figure 3.9. A row may consist of only zeroes. If a row has any nonzero elements, the rightmost of these must be 1. The signs alternate if we read the $(2k-1)$th row from right to left, and then continue to read the $2k$th row from left to right. Hence the sum of the elements in a double row must be 1.

$$\left.\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}\right)$$

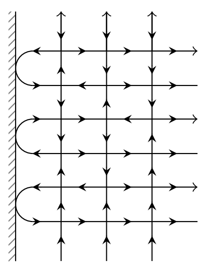
Figure 3.9: A U-turn ASM.

A VSASM is an $n \times n$ ASM, whose entries are mirrored in the middle column, see Figure 3.10. The symmetry forces n to be odd and the middle column to consist of alternating 1's and -1's, without any 0's. Since the middle column is always the same, we can ignore it and only consider the pattern on the right half. The top row of this half matrix consists of 0's so we can ignore it as well. In this way, a VSASM can be viewed as a special case of UASMs, see Figure 3.10.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & -1 & 1 & \mathbf{-1} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 1 & -1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & -1 & 1 & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & -1 & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Figure 3.10: A vertically symmetric ASM. The boldface black part shows the corresponding UASM.

There is a bijection between the UASMs and the states of the 6V model with DWBC and a (diagonal) reflecting end. In the 6V model with DWBC and a reflecting end, consider only the $c_\pm$ vertices. As in the 6V model with DWBC but without reflecting end, the vertices with weight c_+ and c_- must alternate. Look at a double row. On the upper part of each double row, the rightmost $c_\pm$ vertex must be a c_-, if the row has any $c_\pm$ vertices at all. On the lower part of each double row, the rightmost $c_\pm$ vertex must be a c_+, if the row has any $c_\pm$ vertices at all, see further the proof of Lemma 3.1 in Paper I. Furthermore the leftmost $c_\pm$ vertices (if any at all) can only be c_+ vertices at a k_+ turn, or c_- vertices at a k_- turn. This suggests that in the bijection to the UASMs, c_- in the upper part of a double row corresponds to 1, and c_+ corresponds to -1. On the lower part of a double row, c_+ corresponds to 1, and c_- corresponds to -1. In this way the signs will alternate along a double row, see Figure 3.11. The states where all turns are negative correspond to the VSASMs.



$$\longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 3.11: The bijection between a state of the 6V model with DWBC and one reflecting end and a UASM.

In Paper III we consider a sort of generalized ASMs, by loosening the conditions that have to hold. Consider matrices of size $2n \times m$, $m \leq n$, consisting of elements 0, -1 and 1, such that the nonzero elements alternate in signs vertically and horizontally. Connect the rows pairwise on the left edge to form a double row. The sum of the elements of each column is 1, as for ASMs and UASMs. Horizontally however, we change the conditions. A double row may consist of only zeroes. If a row has any nonzero elements, the rightmost of these must be 1. In this way, the objects are UASMs from the right, but at the turn, the signs of the nonzero elements do not necessarily need to switch when going from one row to the next, see Figure 3.12. The sum of the entries in a double row have to be 0 or 1. Equivalently, a double row can not consist of two rows both having 1 as their leftmost nonzero element (however both of the leftmost nonzero elements could be -1). In the case $m = n$, the sum of the elements in each double row is forced to be 1, and we get the UASMs.

$$
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

Figure 3.12: The three matrices on the left are generalized UASMs of the type we consider. The rightmost matrix is not allowed in our setting.

3.6 The XXZ spin chain

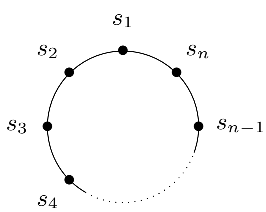


Figure 3.13: A spin chain of length n with spins s_j and periodic boundary conditions.

Consider a one-dimensional lattice with n sites. Let $V_j = \mathbb{C}^2$ be a vector space with the standard basis e_+ and e_-, belonging to the jth site of the chain. Associate $\uparrow$ (spin up) with e_+, and $\downarrow$ (spin down) with e_-. Let $\mathcal{H}_n = V_1 \otimes \cdots \otimes V_n$ be the Hilbert space of the whole spin chain, with tensor products of e_+ and e_- as basis. We denote the basis elements $e_{s_1} \otimes \cdots \otimes e_{s_n}$ with bra-ket notation as $|s_1 \dots s_n\rangle$ where $s_j \in \{\uparrow, \downarrow\}$. Let

$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.24}
$$

be the Pauli matrices with respect to the standard basis, such that $\sigma_z e_\pm = \pm e_\pm$. Let $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, and $\sigma_z^{(j)}$ be the Pauli matrices acting on the jth site of the spin chain, e.g. $\sigma_x^{(j)} = \mathrm{Id}^{\otimes(j-1)} \otimes \sigma_x \otimes \mathrm{Id}^{\otimes(n-j)}$, where Id is the 2×2 identity matrix. Impose periodic boundary conditions, such that $\sigma_x^{(n+j)} = \sigma_x^{(j)}$, $\sigma_y^{(n+j)} = \sigma_y^{(j)}$, and $\sigma_z^{(n+j)} = \sigma_z^{(j)}$. We can think of this periodic spin chain as n quantum particles of spin $1/2$ on a circle, as in Figure 3.13.

The *XXZ spin chain* is defined by the Hamiltonian

$$H_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{j=1}^{n}\left(\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)} + \Delta\sigma_z^{(j)}\sigma_z^{(j+1)}\right). \tag{3.25}$$

For $\Delta = \frac{q+q^{-1}}{2}$, the XXZ-Hamiltonian H_{XXZ} commutes with the transfer matrix t of the homogeneous 6V model (i.e. $\mu_i = \mu_j$ for all $1 \leq i, j \leq n$).

3.6.1 Supersymmetry

We do not use supersymmetry in this thesis, but we mention it since it gives a physical motivation to why the case $\Delta = -1/2$ is important. Consider $\Delta = -1/2$ (given by $\eta = -2/3$). Let T_n be the translation operator acting on the basis elements in $\mathcal{H}_n$ as

$$T_n \left|s_1 \ldots s_{n-1} s_n\right\rangle = \left|s_n s_1 \ldots s_{n-1}\right\rangle. \tag{3.26}$$

Consider a state with $T_n \left|\Psi\right\rangle = t_n \left|\Psi\right\rangle$. Since $T_n^n = \mathrm{Id}$, the eigenvalue t_n is an nth root of unity. Now introduce operators $q_j : \mathcal{H}_n \to \mathcal{H}_{n+1}$, $0 \leq j \leq n$, which act on site j as

$$q_j \left|s_1 \cdots s_{j-1} \uparrow s_{j+1} \cdots s_n\right\rangle = 0, \tag{3.27}$$

$$q_j \left|s_1 \cdots s_{j-1} \downarrow s_{j+1} \cdots s_n\right\rangle = (-1)^{j-1} \left|s_1 \cdots s_{j-1} \uparrow\uparrow s_{j+1} \cdots s_n\right\rangle, \tag{3.28}$$

for $1 \leq j \leq n$, i.e. basis elements with $\uparrow$ at site j are canceled by q_j, whereas a $\downarrow$ at site j is changed to $\uparrow\uparrow$. Because of this, the spins $s_{j+1} \cdots s_n$ are shifted one step to the right. Because of the periodic boundary conditions, we also need to introduce

$$q_0 \left|s_1 \cdots s_{n-1} \uparrow\right\rangle = 0, \tag{3.29}$$

$$q_0 \left|s_1 \cdots s_{n-1} \downarrow\right\rangle = -\left|\uparrow s_1 \cdots s_{n-1} \uparrow\right\rangle. \tag{3.30}$$

Like q_n, this operator acts on the last site of the spin chain. It cancels an element with $\uparrow$ on the last site, whereas a $\downarrow$ is changed to a pair $\uparrow\uparrow$ on site $n+1$ and 1.

Let W_n be the subspace of $\mathcal{H}_n$ consisting of eigenvectors $\left|\phi\right\rangle$ for T_n with eigenvalue $(-1)^{n+1}$, i.e.

$$W_n = \{\left|\phi\right\rangle \in \mathcal{H}_n : T_n \left|\phi\right\rangle = (-1)^{n+1} \left|\phi\right\rangle\}. \tag{3.31}$$

Define an operator $Q_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$, called a supercharge, by

$$Q_n = \begin{cases} \left(\frac{n}{n+1}\right)^{1/2} \sum_{j=0}^{n} q_j & \text{on } W_n, \\ 0 & \text{on } \mathcal{H}_n \setminus W_n. \end{cases} \tag{3.32}$$

These operators decrease the number of down pointing spins. Now define the adjoint supercharge $Q_n^\dagger : \mathcal{H}_{n+1} \to \mathcal{H}_n$ as the Hermitian conjugate of Q_n by $\langle \phi_n | \, Q_n^\dagger \, | \phi_{n+1} \rangle = \langle \phi_{n+1} | \, Q_n \, | \phi_n \rangle^*$, where $|\phi_n\rangle \in \mathcal{H}_n$ and $|\phi_{n+1}\rangle \in \mathcal{H}_{n+1}$. Both Q_n and $Q_n^\dagger$ are "nilpotent", in the sense that $Q_{n+1}Q_n = 0$ and $Q_n^\dagger Q_{n+1}^\dagger = 0$. Restricted to W_n, the XXZ-Hamiltonian can be written in terms of an "anticommutator" of the supercharges and adjoint supercharges,

$$H_{\text{XXZ}} = Q_{n-1}Q_{n-1}^\dagger + Q_n^\dagger Q_n + E_0, \tag{3.33}$$

where $E_0 = -3n/4$ is the ground state eigenvalue of H_{XXZ}. In this case, the spin chain is called *supersymmetric*. For further details, see [14, 18].

3.6.2 Components of the ground state eigenvectors

Consider spin chains of odd length $n = 2k + 1$ with $\Delta = -1/2$. In this case, the eigenvalues of the XXZ-Hamiltonian are (at least) doubly degenerate. In particular, there are two ground state eigenvectors

$$H_{\text{XXZ}} \, |\Psi_\pm\rangle = E_0 \, |\Psi_\pm\rangle \, . \tag{3.34}$$

Because of parity symmetry, it is enough to consider the eigenvector $|\Psi_-\rangle$, which is the ground state eigenvector for which all nonzero components have an odd number of down spins. The number of down spins is k for odd k, and $k+1$ for even k. Denote the coefficients of $|s_1 \ldots s_n\rangle$ in the components of $|\Psi_-\rangle$ by $\Psi_{s_1 \cdots s_n}$, where $s_j \in \{\uparrow, \downarrow\}$. Normalize $|\Psi_-\rangle$ so that

$$\Psi_{\underbrace{\uparrow \ldots \uparrow}_{k+1}\underbrace{\downarrow \ldots \downarrow}_{k}} = 1, \text{ for odd } k, \quad \text{and} \quad \Psi_{\underbrace{\uparrow \ldots \uparrow}_{k}\underbrace{\downarrow \ldots \downarrow}_{k+1}} = 1, \text{ for even } k. \tag{3.35}$$

This turns out to be the smallest nonzero component. Razumov and Stroganov [29] conjectured that the largest component,

$$\Psi_{\uparrow\uparrow\downarrow\uparrow\downarrow\cdots\uparrow\downarrow}, \text{ for odd } k, \quad \text{and} \quad \Psi_{\uparrow\downarrow\cdots\uparrow\downarrow\uparrow\downarrow\downarrow}, \text{ for even } k, \tag{3.36}$$

is given by A_k which is the number of ASMs of size $k \times k$ (see Theorem 3.6). Furthermore they stated conjectures about the sum of the components and the sum of the squares of the components. All these numbers seem to be counting

ASMs in different ways. These conjectures have now been proven [33, 12].

The study of related problems eventually led to the famous Razumov-Stroganov conjecture [30], which was proven by Cantini and Sportiello [10]. It gives a combinatorial interpretation of the ground state for the XXZ spin chain of even length with twisted periodic boundary conditions by relating it to so called fully packed loops, see e.g. [44].

4 The eight-vertex SOS model

In this section, we focus on the eight-vertex (8V) solid-on-solid (SOS) model, which is a generalization of the 6V model. In the first two sections, we give an algebraic description of the 8VSOS model with domain wall boundary conditions (DWBC) and a reflecting end, and summarize the proof of Filali's determinant formula for the partition function. Then we discuss the connections to the three-color model in Section 4.3.

4.1 Algebraic description of the 8VSOS model with a reflecting end

Let $p = e^{2\pi i\tau}$ and $q = e^{2\pi i\eta}$, where τ and η are fixed parameters with $\mathrm{Im}(\tau) > 0$ and $\eta \notin \mathbb{Z} + \tau\mathbb{Z}$. By q^x we mean $e^{2\pi i\eta x}$, and when we write $p^{1/2}$, we mean $p^{1/2} = e^{\pi i\tau}$. Define the *theta function*

$$\vartheta(x, p) = \prod_{j=0}^{\infty}(1 - p^j x)(1 - p^{j+1}/x). \tag{4.1}$$

We will often suppress the p and write $\vartheta(x) := \vartheta(x, p)$, and write out the second parameter only when it is not just p. Define the short hand notation

$$[x] = q^{-x/2}\vartheta(q^x). \tag{4.2}$$

When $p = 0$, we get $\vartheta(x) = 1 - x$, and $[x] = q^{-x/2} - q^{x/2}$ as in (3.1) (up to a constant). There are several useful identities for the theta functions, including

$$\vartheta(x^2) = \vartheta(x)\vartheta(-x)\vartheta(p^{1/2}x)\vartheta(-p^{1/2}x), \tag{4.3}$$

and the addition rule

$$\vartheta(x_1x_3)\vartheta(x_1/x_3)\vartheta(x_2x_4)\vartheta(x_2/x_4) - \vartheta(x_1x_4)\vartheta(x_1/x_4)\vartheta(x_2x_3)\vartheta(x_2/x_3) \\ = \frac{x_2}{x_3}\vartheta(x_1x_2)\vartheta(x_1/x_2)\vartheta(x_3x_4)\vartheta(x_3/x_4), \quad (4.4)$$

where x, x_1, x_2, x_3, x_4 are arbitrary.

Consider the 8VSOS model on a $2n \times n$ lattice where the horizontal lines are connected pairwise at the left boundary, as in Figure 3.5. As in Section 3.1, assign a two-dimensional vector space V to each line. Given parameters $\rho, \lambda \in \mathbb{C}$, define operators $R(\lambda, q^\rho) \in \mathrm{End}(V \otimes V)$ by

$$R(\lambda, q^\rho)(e_\alpha \otimes e_\beta) = \sum_{\alpha+\beta=\alpha'+\beta'} w\Big({}_{\alpha}{}^{\beta'}_{\beta}{}\alpha'\Big)(\lambda, q^\rho) e_{\alpha'} \otimes e_{\beta'}, \quad (4.5)$$

where $w\Big({}_{\alpha}{}^{\beta'}_{\beta}{}\alpha'\Big)(\lambda, q^\rho)$ is one of the weights $a_\pm(\lambda, q^\rho)$, $b_\pm(\lambda, q^\rho)$ or $c_\pm(\lambda, q^\rho)$ belonging to a vertex with spins $\alpha, \beta, \alpha', \beta'$ on the surrounding edges as in Figure 2.1. Similarly as for the 6V model, it is possible to define this operator in a way such that it satisfies the *dynamical Yang–Baxter equation* (DYBE) on $V_1 \otimes V_2 \otimes V_3$ (where each V_i is a copy of V) [13], i.e.

$$R_{12}(\lambda_1 - \lambda_2, q^{\rho-\sigma_z^{(3)}})R_{13}(\lambda_1 - \lambda_3, q^\rho)R_{23}(\lambda_2 - \lambda_3, q^{\rho-\sigma_z^{(1)}}) \\ = R_{23}(\lambda_2 - \lambda_3, q^\rho)R_{13}(\lambda_1 - \lambda_3, q^{\rho-\sigma_z^{(2)}})R_{12}(\lambda_1 - \lambda_2, q^\rho), \quad (4.6)$$

where R_{ij} are defined similarly as in Section 3.1, and where $\sigma_z^{(i)}$ is the third Pauli operator acting on the basis vectors of the ith space, i.e. $\sigma_z e_\pm = \pm e_\pm$. The DYBE depends on the spectral parameters as well as the heights, see Figure 4.1.

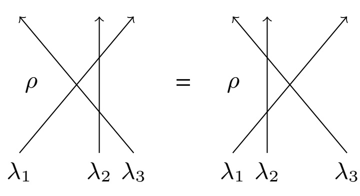


Figure 4.1: The dynamical Yang–Baxter equation.

Fix a parameter $\zeta \in \mathbb{C}$. To describe the reflecting boundary, define a diagonal operator $K(\lambda, q^\rho, q^\zeta) \in \mathrm{End}(V)$. Similar to how we did for the 6V model with a reflecting end, we can define the matrix $K(\lambda, q^\rho, q^\zeta)$ such that it satisfies the *dynamical reflection equation* on $V_0 \otimes V_{0'}$, i.e. [38, 8]

$$R_{00'}(\lambda-\lambda',q^\rho)K_0(\lambda,q^\rho,q^\zeta)R_{0'0}(\lambda+\lambda',q^\rho)K_{0'}(\lambda',q^\rho,q^\zeta)$$
$$=K_{0'}(\lambda',q^\rho,q^\zeta)R_{00'}(\lambda+\lambda',q^\rho)K_0(\lambda,q^\rho,q^\zeta)R_{0'0}(\lambda-\lambda',q^\rho), \quad (4.7)$$

where $K_0(\lambda,q^\rho,q^\zeta)=K(\lambda,q^\rho,q^\zeta)\otimes \mathrm{Id}$ and $K_{0'}(\lambda,q^\rho,q^\zeta)=\mathrm{Id}\otimes K(\lambda,q^\rho,q^\zeta)$, see Figure 4.2.

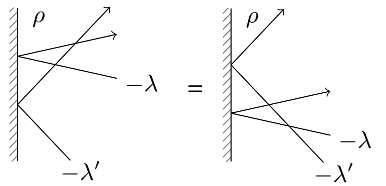


Figure 4.2: The dynamical reflection equation for the matrix $K(\lambda,q^\rho,q^\zeta)$.

The weights of the 8VSOS model are given in terms of theta functions. The 8VSOS model with a reflecting end is given by the following solutions of (4.6) [5] and (4.7) [8]:

$$R(\lambda,q^\rho)=\begin{pmatrix} a_+(\lambda,q^\rho) & 0 & 0 & 0\\ 0 & b_+(\lambda,q^\rho) & c_-(\lambda,q^\rho) & 0\\ 0 & c_+(\lambda,q^\rho) & b_-(\lambda,q^\rho) & 0\\ 0 & 0 & 0 & a_-(\lambda,q^\rho)\end{pmatrix}, \quad (4.8)$$

and

$$K(\lambda,q^\rho,q^\zeta)=\begin{pmatrix} k_+(\lambda,q^\rho,q^\zeta) & 0\\ 0 & k_-(\lambda,q^\rho,q^\zeta)\end{pmatrix}, \quad (4.9)$$

where the entries can be parametrized by

$$a_+(\lambda,q^\rho)=a_-(\lambda,q^\rho)=\frac{[\lambda+1]}{[1]},$$
$$b_+(\lambda,q^\rho)=\frac{[\lambda][\rho-1]}{[\rho][1]}, \qquad b_-(\lambda,q^\rho)=\frac{[\lambda][\rho+1]}{[\rho][1]},$$
$$c_+(\lambda,q^\rho)=\frac{[\rho+\lambda]}{[\rho]}, \qquad c_-(\lambda,q^\rho)=\frac{[\rho-\lambda]}{[\rho]},$$
$$k_+(\lambda,q^\rho,q^\zeta)=\frac{[\rho+\zeta-\lambda]}{[\rho+\zeta+\lambda]}, \qquad k_-(\lambda,q^\rho,q^\zeta)=\frac{[\zeta-\lambda]}{[\zeta+\lambda]}. \quad (4.10)$$

These functions correspond to the states as in Figure 2.6 and in Figure 2.11.

Remark 4.1. To keep it simple in Section 2.1, we defined the weights as $w(\lambda,z)$ and $k_\pm(\lambda,z,\zeta)$, without introducing q. Here we have defined these weights as $w(\lambda,q^z)$ and $k_\pm(\lambda,q^z,q^\zeta)$ instead, as it seems better for Paper I and II.

Remark 4.2. In general, $K(\lambda, q^\rho, q^\zeta)$ does not need to be diagonal, but in this thesis we only consider diagonal matrices $K(\lambda, q^\rho, q^\zeta)$.

Remark 4.3. In the trigonometric limit, i.e. letting $p \to 0$, and then $q^\rho \to \infty$, the elliptic weights become essentially the trigonometric weights in (3.4) and (3.16) (some more technicalities are needed to make the relation precise).

The partition function of the 8VSOS model with DWBC and a reflecting end depends on the vertex weights as well as on the boundary weights of the turns. Let $w(\text{vertex})$ be the local weight of a vertex, which depends on the λ_i and μ_j belonging to the lines passing through the vertex and on the heights of the adjacent faces, and let $w(\text{turn})$ be the local weight of a turn, which depends on the λ_i belonging to the line through the turn, the height outside the turn and the boundary parameter ζ. Fix the reference height in the upper left corner to be ρ. Then the partition function can be defined as

$$Z_n(q^{\lambda_1}, \ldots, q^{\lambda_n}, q^{\mu_1}, \ldots, q^{\mu_n}, q^\rho, q^\zeta) = \sum_{\text{states}} \prod_{\text{vertices}} w(\text{vertex}) \prod_{\text{turns}} w(\text{turn}). \tag{4.11}$$

Implicitly the partition function also depends on η and τ.

Remark 4.4. One needs to check that the partition function is well-defined, i.e. that specifying q^{λ_i} and $q^{\lambda_i+1/\eta}$ yield the same results, and likewise for μ_j, ρ and ζ. The weights contain factors $q^{-x/2} = e^{-\pi i \eta x}$, for $x = \lambda_i, \mu_j, \rho$ and ζ, but $q^{-(x+1/\eta)/2} = -q^{-x/2}$, so one needs to be careful. The proof is included in Paper I (Section 2.2).

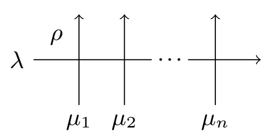


Figure 4.3: The monodromy matrix.

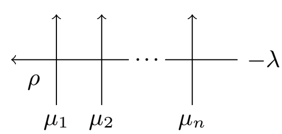


Figure 4.4: The opposite monodromy matrix.

To define the transfer matrix, we need a double row monodromy matrix, which is created as follows. Define the monodromy matrix

$$T_0(\lambda, \mu_1, \ldots, \mu_n, q^\rho) = R_{0n}(\lambda - \mu_n, q^{\rho - \sum_{i=1}^{n-1} \sigma_z^{(i)}}) \cdots R_{01}(\lambda - \mu_1, q^\rho), \tag{4.12}$$

as in Figure 4.3, and the opposite monodromy matrix

$$\overline{T}_0(\lambda, \mu_1, \ldots, \mu_n, q^\rho) = R_{10}(\mu_1 + \lambda, q^\rho) \cdots R_{n0}(\mu_n + \lambda, q^{\rho - \sum_{i=1}^{n-1} \sigma_z^{(i)}}), \tag{4.13}$$

as in Figure 4.4.

Then we define the *double row monodromy matrix* by

$$\mathcal{T}_0(\lambda, \mu_1, \ldots, \mu_n, q^\rho, q^\zeta)$$
$$= T_0(\lambda, \mu_1, \ldots, \mu_n, q^\rho) K(\lambda, q^\rho, q^\zeta) \overline{T}_0(\lambda, \mu_1, \ldots, \mu_n, q^\rho), \quad (4.14)$$

as in Figure 4.5. Commuting double row transfer matrices are then obtained by multiplying the double row monodromy matrix by a second K-matrix associated to a boundary on the right, and then taking the trace over V_0 (see [38] for details).

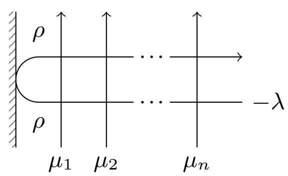


Figure 4.5: The double row monodromy matrix

4.2 Filali's determinant formula

By following the Izergin–Korepin method of Section 3.3, Filali [15] could find a determinant formula for the partition function of the 8VSOS model with DWBC and a reflecting end, i.e. by determining properties which together define the partition function in a unique way, and then suggesting a formula which satisfies the conditions. The following conditions hold true for the partition function of the 8VSOS model with DWBC and a reflecting end.

(i) The partition function of the 8VSOS model with DWBC and a reflecting end is symmetric in the λ_i's and in the μ_j's respectively.

Proof. The symmetry in the μ_j's is proven with the DYBE in the same way as in Lemma 3.2. The alternating orientations on the horizontal lines do not affect the train argument. To prove the symmetry in the λ_i's, we add two extra vertices on the right, as in Figure 4.6. The vertices can then be pulled through each other in a similar way as in Lemma 3.2, using the DYBE and the reflection equation. Because of the boundary conditions and the ice rule, the extra vertices give rise to two extra factors on each side of the equation, see Figure 4.6. Since $a_+(\lambda, q^\rho) = a_-(\lambda, q^\rho)$, the extra factors cancel. □

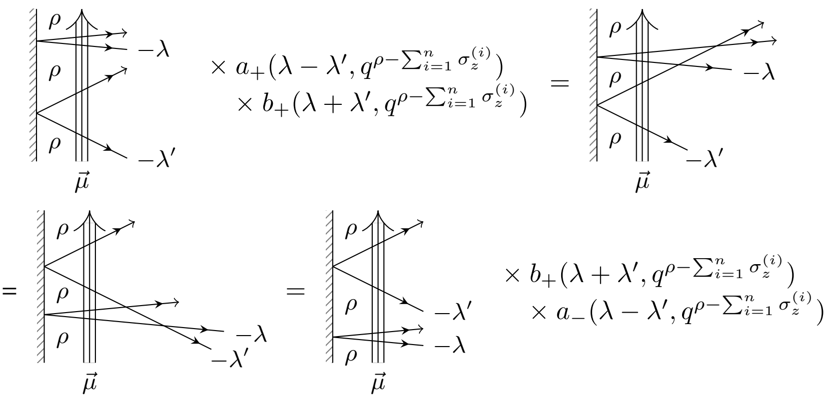


Figure 4.6: The partition function is symmetric in the λ_i's and μ_j's. The triple arrows should be understood as n vertical arrows.

(ii) For $n = 1$, the partition function is just a sum of two terms, since there are only two possible states.

(iii) Specializing $\lambda_n = \mu_n$ in the partition function for a $2n \times n$ lattice yields a recurrence relation for Z_n. In the same way, $\lambda_1 = -\mu_n$ yields another recurrence relation. By the symmetry in (i), any specialization $\lambda_i = \pm\mu_j$ yields a recurrence relation.

Definition 4.5. Fix η and τ. Then f is a theta function of *order* n and *norm* t in the variable x if there exist constants $a_1, \ldots, a_n$ and C with $a_1 + \cdots + a_n = t$, such that $f(x) = C[x - a_1] \cdots [x - a_n]$.

(iv) The normalized partition function

$$\begin{aligned} &\tilde{Z}_n(q^{\lambda_1}, \ldots, q^{\lambda_n}, q^{\mu_1}, \ldots, q^{\mu_n}, q^\rho, q^\zeta) \\ &= \prod_{i=1}^n \frac{[\rho + \zeta + \lambda_i][\rho + \lambda_i]}{[2\lambda_i]} Z_n(q^{\lambda_1}, \ldots, q^{\lambda_n}, q^{\mu_1}, \ldots, q^{\mu_n}, q^\rho, q^\zeta) \end{aligned} \quad (4.15)$$

is a theta function of order $2n - 2$ and norm $n - 1$ as a function of the variable λ_i.

Idea of proof. To find the order, check that each vertex weight has one factor $[\lambda_i - a]$, for some a, which together have order $2n$ in λ_i. Each turn has one such factor in the numerator, and a factor $[\rho + \zeta + \lambda_i]$ or $[\zeta + \lambda_i]$ in the denominator which is canceled by one of the prefactors in the normalized partition function. The denominator of the prefactor together with the turns add 2 to the order. As we explain below, one can prove that the partition function is divisible by $[2\lambda_i]$. We can rewrite $[2\lambda_i]$ with (4.3) and thus realize that each such factor has order 4. All of this sums up to $2n-2$.

To find the norm, and to prove that the partition function is divisible by $[2\lambda_i]$. Filali uses an argument with a change of basis and a Drinfel'd twist. Alternatively, both these properties follow immediately from the so called crossing symmetry of the partition function under transformations $\lambda_i \to -\lambda_i - 1$. A proof of this symmetry is given in [24]. □

Because of (iv), the partition function is completely determined by its values in $2n-2$ independent points: The order yields that it is enough to specify $2n-1$ points, and because of the norm, one of these points is dependent on the others so we need one point less. By (iii), there are enough points $\lambda_i = \pm\mu_j$ for which we can find recurrence relations. Hence by induction, starting at $n=1$, it follows that the partition function is uniquely determined.

In [15], Filali presented a determinant formula for the partition function. The formula can be proven by checking (i) –(iv) above.

Theorem 4.6 (Filali)**.** *The partition function of the 8VSOS model with DWBC and a reflecting end is*

$$\begin{aligned} &Z_n(q^{\lambda_1},\dots,q^{\lambda_n},q^{\mu_1},\dots,q^{\mu_n},q^{\rho},q^{\zeta}) \\ &= [1]^{n-2n^2}\prod_{i=1}^{n}\frac{[2\lambda_i][\zeta-\mu_i][\rho+\zeta+\mu_i][\rho+(2i-n-2)]}{[\zeta+\lambda_i][\rho+\zeta+\lambda_i][\rho+(n-i)]} \\ &\quad\times\frac{\prod_{i,j=1}^{n}[\lambda_i+\mu_j+1][\lambda_i-\mu_j+1][\lambda_i+\mu_j][\lambda_i-\mu_j]}{\prod_{1\le i<j\le n}[\lambda_i+\lambda_j+1][\lambda_i-\lambda_j][\mu_j+\mu_i][\mu_j-\mu_i]}\det_{1\le i,j\le n}K, \end{aligned} \tag{4.16}$$

where

$$K_{ij} = \frac{1}{[\lambda_i+\mu_j+1][\lambda_i-\mu_j+1][\lambda_i+\mu_j][\lambda_i-\mu_j]}. \tag{4.17}$$

4.3 The 8VSOS model and the three-color model

By specializing the parameters in the determinant formula for the partition function of the 6V model with DWBC, Kuperberg found an enumeration of the ASMs. With the same specialization in the partition function of the 8VSOS model with DWBC (no reflecting end), Rosengren [34] found an expression for the partition function of the three-color model with the same boundary conditions.

To get a partition function invariant under translations with $1/\eta$ in the variables λ_i, μ_i and ρ, Rosengren defines the normalized partition function

$$\tilde{Z}(q^{\lambda_1},\dots,q^{\lambda_n},q^{\mu_1},\dots,q^{\mu_n},q^{\rho}) = q^{n\sum_{i=1}^{n}(\lambda_i+\mu_i)/2} \sum_{\text{states}} \prod_{\text{vertices}} w(\text{vertex}), \tag{4.18}$$

with the vertex weights $w(\text{vertex})$ defined as in Section 4.1. Taking out some factors from the vertex weights into a prefactor, it is possible to write the remaining part of the vertex weight as a function depending only on the adjacent faces, and not on the type of vertex. Let ρ be the reference height in the upper left corner of the lattice and define $\tilde{\rho} = q^{\rho}$. Specialize $\eta = -2/3$, $\lambda_i = -1/2$ and $\mu_j = 0$. Define $\omega = e^{2\pi i/3}$. Then the partition function (4.18) specializes to

$$\tilde{Z}(\omega,\dots,\omega,1,\dots,1,\tilde{\rho}) = \omega^{\binom{n+1}{2}} \vartheta(\tilde{\rho}^3,p^3)^{n^2} \sum_{\text{states}} \prod_{\text{vertices}} \frac{1}{\vartheta(\tilde{\rho}\omega^a)\vartheta(\tilde{\rho}\omega^b)\vartheta(\tilde{\rho}\omega^d)}, \tag{4.19}$$

where a, b, d are the heights on three of the four adjacent faces of each vertex. Hence in all states, each face in the interior of the lattice gives rise to three factors $1/\vartheta(\tilde{\rho}\omega^a)$. On the boundaries, each face gives rise to a different number of factors, but because of the boundary conditions, this contribution is the same in all states and one can compute the contribution explicitly. Finally the partition function becomes

$$\tilde{Z}(\omega,\dots,\omega,1,\dots,1,\tilde{\rho}) = \omega^{\binom{n+1}{2}} \frac{\vartheta(\tilde{\rho}^3,p^3)^{n^2+2n+2}\vartheta(\tilde{\rho}\omega^n)}{\vartheta(\tilde{\rho}\omega^2)^2\vartheta(\tilde{\rho}\omega^{n+1})^2} \sum_{\text{states}} \prod_{\text{faces}} \frac{1}{\vartheta(\tilde{\rho}\omega^a)^3}. \tag{4.20}$$

By specifying $t_a = 1/\vartheta(\tilde{\rho}\omega^a)^3$, the sum yields the partition function of the three-color model. As a corollary, Rosengren could find the probability that a random face of a random state of the three-color model has color a. In Paper I, we do the above (except for Rosengren's corollary) for the 8VSOS model with DWBC and a reflecting end.

5 Special polynomials

In Section 3, we saw that the XXZ spin chain for $\Delta = -1/2$ has many connections to combinatorics. When studying similar problems for the XYZ spin chain, special polynomials show up. Many of these polynomials seem to have positive coefficients. This indicates that there should be combinatorial interpretations in the XYZ case as well, but up till now, not much is known about these connections. In Section 5.1 and Section 5.2, we introduce certain polynomials of Bazhanov and Mangazeev. These polynomials have been conjectured to be a special case of Rosengren´s polynomials, which we present in Section 5.3 and Section 5.4.

5.1 Bazhanov's and Mangazeev's polynomials

In order to solve the 8V model, Baxter [3] introduced the Q-operator. Bazhanov and Mangazeev [6] studied the ground state eigenvalue of this operator in the case where $\eta = -2/3$. They found that the eigenvalue can be written in terms of certain polynomials

$$\mathcal{P}_n(x,z) = \sum_{k=0}^{n} r_k^{(n)}(z) x^k, \tag{5.1}$$

where $r_k^{(n)}(z)$, $k = 0, \dots, n$ are polynomials with integer coefficients, normalized by $r_n^{(n)}(0) = 1$. The polynomials $r_k^{(n)}(z)$ are conjectured to have *positive* integer coefficients [7]. They introduced polynomials $s_n(z) = r_n^{(n)}(z)$ and $\overline{s}_n(z) = r_0^{(n)}(z)$ and gave a recurrence relation [6], with initial conditions $s_0(z) = s_1(z) = 1$, which uniquely determines the polynomials $s_n(z)$ for all $n \in \mathbb{Z}$. Furthermore they stated the following conjecture [26].

Conjecture 5.1 (Mangazeev, Bazhanov)**.**

1. *The polynomials* $s_{2k+1}(z^2)$, $k \in \mathbb{Z}$, *factorize over the integers,*

$$s_{2k+1}(z^2) = s_{2k+1}(0) p_k(z) p_k(-z), \tag{5.2}$$

where $p_k(z)$ *are polynomials in* z *with integer coefficients,* $p_k(0) = 1$ *and* $\deg p_k(z) = k(k+1)$. *Note that* $p_{-1}(z) = p_0(z) = 1$.

2. *The polynomials* $p_k(z)$, $k \in \mathbb{Z}$, *have the symmetry*

$$p_k(z) = \left(\frac{1+3z}{2}\right)^{k(k+1)} p_k\left(\frac{1-z}{1+3z}\right). \tag{5.3}$$

3. *The polynomials* $s_{2k}(z^2)$, $k \in \mathbb{Z}$, *factorize over the integers,*

$$s_{2k}(z^2) = c_k (1+3z)^{k(k+1)} p_{-k-1}\left(\frac{z-1}{1+3z}\right) q_{k-1}(z), \tag{5.4}$$

where $q_k(z)$ *are polynomials in* z *with integer coefficients,* $q_k(0) = 1$, $\deg q_k(z) = k(k+1)$, $c_k = 2^{-k(k+2)}$, *for* $k \geq 0$, *and* $c_k = 2^{-k^2}(2/3)^{2k+1}$, *for* $k < 0$.

4. *The polynomials* $q_k(z)$, $k \in \mathbb{Z}$, *have the symmetry*

$$q_k(z) = \left(\frac{1+3z}{2}\right)^{k(k+1)} q_k\left(\frac{z-1}{1+3z}\right). \tag{5.5}$$

The function $s_{2k}(z^2)$ in (5.4) is an even function. It follows from the symmetry in (5.3) that $q_k(z)$ is also even, i.e. $q_k(z) = q_k(-z)$. The polynomials $q_k(z)$ and $p_k(z)$ also seem to have positive coefficients for non-negative k. They were introduced in order to state results about the ground state eigenvectors of the XYZ-Hamiltonian for spin chains of odd length (see Conjecture 5.2).

5.2 The XYZ spin chain

As in Section 3.6, consider a periodic spin chain of length n. The *XYZ spin chain* is defined by the XYZ-Hamiltonian

$$H_{\text{XYZ}} = -\frac{1}{2}\sum_{j=1}^{n}\left(J_x \sigma_x^{(j)}\sigma_x^{(j+1)} + J_y \sigma_y^{(j)}\sigma_y^{(j+1)} + J_z \sigma_z^{(j)}\sigma_z^{(j+1)}\right). \tag{5.6}$$

A special case of the XYZ-Hamiltonian is when

$$J_x J_y + J_y J_z + J_z J_x = 0, \tag{5.7}$$

which can, up to normalization, be parametrized by rational functions of one parameter,

$$J_x = \frac{2(1+\psi)}{\psi^2+3}, \quad J_y = \frac{2(1-\psi)}{\psi^2+3}, \quad \text{and} \quad J_z = \frac{\psi^2-1}{\psi^2+3}. \tag{5.8}$$

For $\psi = 0$, we recover the Hamiltonian of the supersymmetric XXZ spin chain with $\Delta = -1/2$. The XYZ-Hamiltonian commutes with the transfer matrix of the 8V model for appropriate values of the parameters.

The XYZ-Hamiltonian is supersymmetric when (5.8) holds. The supercharges for the XYZ spin chain are defined in a similar way as for the XXZ spin chain in Section 3.6, but now the q_j's depend on the parameter ψ, and are defined by

$$\begin{aligned} q_j \left|s_1 \cdots s_{j-1} \downarrow s_{j+1} \cdots s_n\right\rangle = (-1)^{j-1} \big(&\left|s_1 \cdots s_{j-1} \uparrow\uparrow s_{j+1} \cdots s_n\right\rangle \\ &- \psi \left|s_1 \cdots s_{j-1} \downarrow\downarrow s_{j+1} \cdots s_n\right\rangle \big), \end{aligned} \tag{5.9}$$

$$q_0 \left|s_1 \cdots s_{n-1} \downarrow\right\rangle = - \left(\left|\uparrow s_1 \cdots s_{n-1} \uparrow\right\rangle - \psi \left|\downarrow s_1 \cdots s_{n-1} \downarrow\right\rangle \right). \tag{5.10}$$

Again $\psi = 0$ corresponds to the XXZ case. Because of the extra component for each q_j, the Q_n's do not decrease the number of negative spins in the XYZ case, see further [18].

For odd $n = 2k+1$, the eigenvalues of the XYZ-Hamiltonian are (at least) doubly degenerate. There are two ground state eigenvectors

$$H_{\text{XYZ}} \left|\Psi_\pm\right\rangle = E_0 \left|\Psi_\pm\right\rangle. \tag{5.11}$$

Because of parity symmetry, it is enough to consider $|\Psi_-\rangle$, which is the ground state eigenvector for which all nonzero components have an odd number of down spins. As in the XXZ case, denote the coefficients of its components as $\Psi_{s_1 \cdots s_n}$, where $s_j \in \{\uparrow, \downarrow\}$. Normalize $|\Psi_-\rangle$ so that

$$\Psi_{\underbrace{\uparrow \cdots \uparrow}_{k+1} \underbrace{\downarrow \cdots \downarrow}_{k}} \Big|_{\psi=0} = 1, \text{ for odd } k, \quad \text{and} \quad \Psi_{\underbrace{\uparrow \cdots \uparrow}_{k} \underbrace{\downarrow \cdots \downarrow}_{k+1}} \Big|_{\psi=0} = 1, \text{ for even } k. \tag{5.12}$$

Mangazeev and Bazhanov [26] (see also [32]) stated several conjectures about the components of this ground state eigenvector.

Conjecture 5.2 (Mangazeev, Bazhanov)**.**

1. *The component of the eigenvector* $|\Psi_-\rangle$ *with exactly one spin down is given by*

$$\Psi_{\uparrow\uparrow\ldots\uparrow\uparrow\downarrow} = \frac{1}{n}\psi^{k(k-1)/2}\overline{s}_k(1/\psi^2), \tag{5.13}$$

for $n = 2k+1$.

2. *The component of the eigenvector* $|\Psi_-\rangle$ *with all spins down is given by*

$$\Psi_{\downarrow\downarrow\ldots\downarrow\downarrow} = \psi^{k(k-1)/2}s_k(1/\psi^2), \tag{5.14}$$

for $n = 2k+1$.

3. *Let* $A_k(\psi)$ *be the components of* $|\Psi_-\rangle$ *with alternating spins,*

$$A_k(\psi) = \begin{cases} \Psi_{\uparrow\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow}, & \text{for odd } k, \\ \Psi_{\uparrow\downarrow\ldots\uparrow\downarrow\uparrow\downarrow\downarrow}, & \text{for even } k. \end{cases} \tag{5.15}$$

The components of $|\Psi_-\rangle$ *with alternating spins is given by*

$$A_{2m}(\psi) = 2^{m(2-m)}(3+\psi)^{m(m-1)}\psi^{m(m-1)}p_{m-1}\left(\frac{1-\psi}{3+\psi}\right)q_{m-1}(1/\psi), \tag{5.16}$$

$$A_{2m+1}(\psi) = 2^{-m^2}(3+\psi)^{m(m+1)}\psi^{m(m-1)}p_{m-1}\left(\frac{1-\psi}{3+\psi}\right)q_{m-1}(1/\psi). \tag{5.17}$$

Progress on these conjectures has been made by Hagendorf (private correspondence).

5.3 Rosengren's special polynomials

Rosengren [36, 37] introduced a general family of polynomials T, which seems to contain the polynomials s_n, $\overline{s}_n$, p_n, and q_n of Bazhanov and Mangazeev as special cases. Define

$$G(x,y) = (\psi+2)xy(x+y) + \psi(2\psi+1)(x+y) - 2(\psi^2+3\psi+1)xy - \psi(x^2+y^2), \tag{5.18}$$

let $\Delta(x_1,\dots,x_n)=\prod_{1\le i<j\le n}(x_j-x_i)$, and define

$$T(x_1,\dots,x_{2n})=\frac{\prod_{i,j=1}^n G(x_j,x_{n+i})}{\Delta(x_1,\dots,x_n)\Delta(x_{n+1},\dots,x_{2n})}\det_{1\le i,j\le n}\left(\frac{1}{G(x_j,x_{n+i})}\right). \tag{5.19}$$

Up to a prefactor and a variable change, these polynomials are equivalent to Filali's determinant (4.16), see Section 4.1 of Paper I. Recall that $p=e^{2\pi i\tau}$ and $\omega=e^{2\pi i/3}$. The variable change is given by [37]

$$\psi=\psi(\tau)=\frac{\omega^2\vartheta(-1)\vartheta(-p^{1/2}\omega)}{\vartheta(-p^{1/2})\vartheta(-\omega)}, \tag{5.20}$$

$$x(z)=x(z,\tau)=\frac{\vartheta(-p^{1/2}\omega)^2\vartheta(\omega e^{\pm 2\pi iz})}{\vartheta(-\omega)^2\vartheta(p^{1/2}\omega e^{\pm 2\pi iz})}. \tag{5.21}$$

The variables x_j in (5.19) are related to the spectral parameters in (4.10) as

$$x_j=x((\lambda_j-1)/3),\quad x_{n+j}=x(\mu_j/3), \tag{5.22}$$

for $1\le j\le n$.

Now we specialize some of the variables to $\xi_i=x(\gamma_i)$, where

$$\gamma_0=0,\quad \gamma_1=\tau/2,\quad \gamma_2=1/2+\tau/2,\quad \gamma_3=1/2, \tag{5.23}$$

i.e.

$$\xi_0=2\psi+1,\quad \xi_1=\frac{\psi}{\psi+2},\quad \xi_2=\frac{\psi(2\psi+1)}{\psi+2},\quad \xi_3=1, \tag{5.24}$$

and define

$$T_n^{(k_0,k_1,k_2,k_3)}(x_1,\dots,x_m)=T(x_1,\dots,x_m,\boldsymbol{\xi}^{(k_0,k_1,k_2,k_3)}), \tag{5.25}$$

where

$$\boldsymbol{\xi}^{(k_0,k_1,k_2,k_3)}=(\underbrace{\xi_0,\dots,\xi_0}_{k_0},\underbrace{\xi_1,\dots,\xi_1}_{k_1},\underbrace{\xi_2,\dots,\xi_2}_{k_2},\underbrace{\xi_3,\dots,\xi_3}_{k_3}), \tag{5.26}$$

and m and k_i are non-negative integers and $m+\sum_{i=0}^3 k_i=2n$.

Rosengren extends the definition of $T_n^{(k_0,k_1,k_2,k_3)}(x_1,\dots,x_m)$ to negative k_i's as well. This extension can be motivated from the recursion relation (iii) in Section 4.2, which says that specializing $\lambda_i=\pm\mu_j$, for given i and j, in the $2n\times n$ determinant formula yields a determinant of one size smaller. Hence specializing one variable in T_n to $x(\gamma_i)$ and another to $x(\gamma_i+1/3)$ should also reduce n by 1. As we have seen, specializing a variable to $x(\gamma_i)$ corresponds to increasing k_i by 1, so $x(\gamma_i + 1/3)$ must reduce k_i by 1 to add up to a one size smaller determinant T_{n-1}. It turns out that this naive idea does not quite work. One should first apply the operator $(\sigma f)(z) = f(z+1/3) + f(-z+1/3)$ to $-k_i$ variables and then let $z \to \gamma_i$.

More specifically, define

$$T(x_1, \dots, x_k; x_{k+1}, \dots, x_{2n}) = \frac{\left(\mathrm{Id}^{\otimes k} \otimes \hat{\sigma}^{\otimes(2n-k)}\right) \Delta(x_1, \dots, x_{2n}) T(x_1, \dots, x_{2n})}{\Delta(x_1, \dots, x_k)\Delta(x_{k+1}, \dots, x_{2n})}, \quad (5.27)$$

where $\hat{\sigma}$ is a uniformization of σ (see [36] for details). Replacing the operator $\mathrm{Id}^{\otimes k} \otimes \hat{\sigma}^{\otimes(2n-k)}$ by $(-1)^{l(n-k)} \left(\mathrm{Id}^{\otimes k} \otimes \hat{\sigma}^{\otimes(n-k)} \otimes \mathrm{Id}^{\otimes l} \otimes \hat{\sigma}^{\otimes(n-l)}\right)$ in the formula above, we can let $\hat{\sigma}$ act on different variables and we get

$$T(x_1, \dots, x_k, x_{n+1}, \dots, x_{n+l}; x_{k+1}, \dots, x_n, x_{n+l+1}, \dots, x_{2n}) = \frac{\prod_{\substack{1 \le i \le k \\ l+1 \le j \le n}} (x_{n+j} - x_i) \prod_{\substack{k+1 \le i \le n \\ 1 \le j \le l}} (x_i - x_{n+j}) \prod_{i,j=1}^{n} G(x_i, x_{n+j})}{\Delta(x_1, \dots, x_k)\Delta(x_{k+1}, \dots, x_n)\Delta(x_{n+1}, \dots, x_{n+l})\Delta(x_{n+l+1}, \dots, x_{2n})} \det B, \quad (5.28)$$

where

$$B_{i,j} = \begin{cases} \dfrac{1}{G(x_i, x_{n+j})}, & \text{for} \quad 1 \le i \le k, 1 \le j \le l, \\ \dfrac{Q(x_i, x_{n+j})}{(x_{n+j} - x_i) G(x_i, x_{n+j})}, & \text{for} \quad 1 \le i \le k, l+1 \le j \le n, \\ \dfrac{Q(x_{n+j}, x_i)}{(x_i - x_{n+j}) G(x_i, x_{n+j})}, & \text{for} \quad k+1 \le i \le n, 1 \le j \le l, \\ \dfrac{R(x_i, x_{n+j})}{G(x_i, x_{n+j})}, & \text{for} \quad k+1 \le i \le n, l+1 \le j \le n, \end{cases} \quad (5.29)$$

and

$$\begin{aligned} Q(x,y) &= y(y - 2\psi - 2\psi - 1)((\psi+2)y - 3\psi) \\ &\quad - x((\psi+2)y - \psi)(2\psi + 1 - 3y), \quad (5.30) \\ R(x,y) &= 3(\psi+2)^2 x^2 y^2 + \psi(\psi+2)(2\psi+1)(x^2+y^2) \\ &\quad - 2(\psi^2 + 4\psi + 1)((\psi+2)xy + \psi(\psi+1))(x+y) \\ &\quad + 4(\psi^4 + 4\psi^3 + 8\psi^3 + 4\psi + 1)xy + 3\psi^2(2\psi+1)^2. \quad (5.31) \end{aligned}$$

Now to define $T_n^{(k_0,k_1,k_2,k_3)}$ for general k_j, we specialize some of the variables in (5.28) to ξ_j. Let $k_j^+ = \max(k_j, 0)$ and $k_j^- = \max(-k_j, 0)$. Then for $n \in \mathbb{Z}$, $k_j \in \mathbb{Z}$, $0 \leq j \leq 3$, and $m = 2n - \sum_{j=0}^3 k_j \geq 0$, define

$$T_n^{(k_0,k_1,k_2,k_3)}(x_1,\dots,x_m) = \frac{(-1)^{\binom{|k^-|}{2}} T(x_1,\dots,x_m,\boldsymbol{\xi}^{(k_0^+,k_1^+,k_2^+,k_3^+)};\boldsymbol{\xi}^{(k_0^-,k_1^-,k_2^-,k_3^-)})}{2^{|k^-|}\prod_{i,j=0}^3 G(\xi_i,\xi_j)^{k_i^- k_j^+}\prod_{j=1}^m\prod_{i=0}^3 G(x_j,\xi_i)^{k_i^-}}, \quad (5.32)$$

where $|k^-| = \sum_{j=0}^3 k_j^-$.

In the case where all x_i's are one of the ξ_j's, the only dependence on a variable is that the ξ_j's depend on ψ. Therefore we also define

$$t^{(k_0,k_1,k_2,k_3)}(\psi) = T(\underbrace{\xi_0,\dots,\xi_0}_{k_0\text{ times}},\underbrace{\xi_1,\dots,\xi_1}_{k_1\text{ times}},\underbrace{\xi_2,\dots,\xi_2}_{k_2\text{ times}},\underbrace{\xi_3,\dots,\xi_3}_{k_3\text{ times}}), \quad (5.33)$$

with $k_j \in \mathbb{Z}$ and $\sum_{j=0}^3 k_j = 2n$.

The polynomials $T(x_1,\dots,x_{2n})$ possess many symmetries. In Paper I and Paper II, the following symmetries ([37, Proposition 2.2]) are used together with (5.39) and (5.38) below, to get expressions for q_n and p_n in terms of $t^{(2n+2,0,0,0)}(\psi)$ and $t^{(2n+1,0,0,-1)}(\psi)$ respectively.

Proposition 5.3. *The polynomials $t^{(k_0,k_1,k_2,k_3)}(\psi)$ have the symmetries*

$$t^{(k_0,k_1,k_2,k_3)}(\psi) = \left(\frac{\psi+2}{\psi(2\psi+1)}\right)^{n(n-1)}\prod_{j=0}^3 \xi_j^{k_j(n-1)} t^{(k_1,k_0,k_3,k_2)}(\psi) \quad (5.34)$$

$$= \left(\frac{\psi-1}{\psi+2}\right)^{n(n-1)} t^{(k_2,k_1,k_0,k_3)}(-\psi-1). \quad (5.35)$$

5.4 Specializations of Rosengren's polynomials

Zinn-Justin [45] studied polynomials equivalent to Rosengren's polynomials T. He noticed that Bazhanov's and Mangazeev's polynomials seem to be given by specializing the variables in a determinant equivalent to T (5.19). In terms of Rosengren's functions, Bazhanov's and Mangazeev's polynomials are related to $t^{(k_0,k_1,k_2,k_3)}(\psi)$ by [37]

$$\left(\frac{(\psi+2)(2\psi+1)}{2}\right)^{\lfloor n^2/4\rfloor} s_n\left(\frac{\psi}{(\psi+2)(2\psi+1)}\right)$$
$$= \frac{(-1)^{\lfloor n/2\rfloor}\left(\psi/2+1\right)^{n(n-1)-\lfloor (n-1)^2/4\rfloor}}{\psi^{n(n-1)}(\psi+1)^{n(n-1)}(2\psi+1)^{\lfloor (n-1)^2/4\rfloor}} t^{(n,n,0,0)}(\psi), \quad (5.36)$$

$$\left(\frac{(\psi+2)(2\psi+1)}{2}\right)^{\lfloor n^2/4\rfloor} \bar{s}_n\left(\frac{\psi}{(\psi+2)(2\psi+1)}\right)$$
$$= \frac{(-1)^{\lfloor n/2\rfloor+1}2^{n-1}\left(\psi/2+1\right)^{n^2-1-\lfloor (n-1)^2/4\rfloor}}{\psi^{n^2-1}(\psi+1)^{n(n-1)}(2\psi+1)^{\lfloor (n-1)^2/4\rfloor}} t^{(n,n,1,-1)}(\psi), \quad (5.37)$$

$$p_n\left(\frac{1}{2\psi+1}\right) = \frac{(-1)^n C_n(\psi+2)^{n^2-n-1}}{\psi^{n^2-2n-1}(\psi+1)^{n(n-1)}(2\psi+1)^{n^2+n+1}} t^{(-1,2n+1,0,0)}(\psi), \quad (5.38)$$

where $C_n = 2^n$ for $n \geq 0$ and $C_n = 3^{n+1}/2^{n+2}$ for $n \leq -1$,

$$q_n\left(\frac{1}{2\psi+1}\right) = D_n\left(\frac{\psi+2}{\psi(\psi+1)(2\psi+1)}\right)^{n(n+1)} t^{(0,2n+2,0,0)}(\psi), \quad (5.39)$$

where $D_n = 2^n$ for $n \geq -1$ and $D_n = 3^{n+2}/2^{2n+3}$ for $n \leq -2$.

That these polynomials are identical to the polynomials of Bazhanov and Mangazeev defined in the beginning of Section 5.1 is still a conjecture. In Paper I and II, we take (5.38) and (5.39) as the definitions of the polynomials p_n and q_n. They can also be defined via recurrence relations [45]. Let $a = 1 - 1/z^2$. Since $q_k(z)$ is even, we can consider $\tilde{q}_k(a) = q_k(z)$. From [45], we get the recurrence relation

$$D_0\tilde{q}_{k-2}(a)\tilde{q}_k(a) = D_1(\tilde{q}_{k-1}(a))^2 + D_2\tilde{q}_{k-1}(a)\tilde{q}'_{k-1}(a)$$
$$+ D_3(\tilde{q}'_{k-1}(a))^2 + D_4\tilde{q}_{k-1}(a)\tilde{q}''_{k-1}(a), \quad (5.40)$$

for $k \in \mathbb{N}$, with initial conditions $\tilde{q}_{-1}(a) = \tilde{q}_0(a) = 1$, and

$$D_0 = 2(a-1)a(4k-1)(4k+1)^2(4k+3),$$
$$D_1 = 3\left(a^2(4k+1)(40k^3+31k^2-1) - 2a(316k^4+320k^3+75k^2-9k-2)\right.$$
$$\left. + 32(k-1)k(2k+1)(4k+1)\right),$$
$$D_2 = -(a-1)(a+8)\left(a^2(32k^2+12k+1) + 4a(92k^2+36k+7)\right.$$
$$\left. - 32(8k^2+6k+1)\right),$$
$$D_3 = 2(a-1)^2a(a+8)^2(4k-1)(4k+3),$$

$$D_4 = -2(a-1)^2 a(a+8)^2(4k+1)^2. \tag{5.41}$$

The first few polynomials are

$$\begin{aligned}
q_{-1}(z) &= q_0(z) = 1,\\
q_1(z) &= 1+3z^2,\\
q_2(z) &= 1+8z^2+29z^4+26z^6,\\
q_3(z) &= 1+15z^2+112z^4+518z^6+1257z^8+1547z^{10}+646z^{12},\\
q_4(z) &= 1+24z^2+291z^4+2338z^6+13524z^8+54474z^{10}+150472z^{12}\\
&\quad +276678z^{14}+312195z^{16}+192694z^{18}+45885z^{20}.
\end{aligned} \tag{5.42}$$

The polynomials $p_k(z)$ are not even, but we can still define them via a function $\tilde{p}_k(a)$, where

$$p_k(z) = \left(\frac{z+1}{2}\right)^{k(k+1)} \tilde{p}_k\left(\frac{8z(1-z)}{(1+z)^2}\right). \tag{5.43}$$

The recurrence relation for $\tilde{p}_k(a)$ is given by [45]

$$\begin{aligned}
D_0\tilde{p}_{k-2}(a)\tilde{p}_k(a) &= D_1(\tilde{p}_{k-1}(a))^2 + D_2\tilde{p}_{k-1}(a)\tilde{p}'_{k-1}(a)\\
&\quad + D_3(\tilde{p}'_{k-1}(a))^2 + D_4\tilde{p}_{k-1}(a)\tilde{p}''_{k-1}(a),
\end{aligned} \tag{5.44}$$

for $k \in \mathbb{N}$, with initial conditions $\tilde{p}_{-1}(a) = \tilde{p}_0(a) = 1$, and

$$\begin{aligned}
D_0 &= 4a(4k-1)^2(4k-3)(4k+1),\\
D_1 &= a^3k(4k^3-12k^2+9k-1)\\
&\quad -6a^2(168k^4-172k^3+k^2+21k-3)\\
&\quad +12a(284k^4-232k^3-17k^2+28k-3)\\
&\quad +128k(k-1)^2(4k-1),\\
D_2 &= 2(a-1)(a+8)\left(a^2(24k^2-4k+1)+4a(76k^2-24k-1)\right.\\
&\quad \left.-64k(4k-1)\right),\\
D_3 &= -4(a-1)^2a(a+8)^2(4k-3)(4k+1),\\
D_4 &= 4(a-1)^2a(a+8)^2(4k-1)^2.
\end{aligned} \tag{5.45}$$

The first few polynomials are

$$\begin{aligned}
p_{-1}(z) &= p_0(z) = 1,\\
p_1(z) &= 1+z+2z^2,\\
p_2(z) &= 1+2z+7z^2+10z^3+21z^4+12z^5+11z^6,
\end{aligned}$$

$$\begin{aligned}p_3(z) &= 1 + 3z + 15z^2 + 35z^3 + 105z^4 + 195z^5 + 435z^6 + 555z^7 \\ &\quad + 840z^8 + 710z^9 + 738z^{10} + 294z^{11} + 170z^{12}, \\ p_4(z) &= 1 + 4z + 26z^2 + 82z^3 + 319z^4 + 840z^5 + 2488z^6 + 5572z^7 \\ &\quad + 13524z^8 + 24920z^9 + 48776z^{10} + 72800z^{11} + 114716z^{12} \\ &\quad + 135464z^{13} + 169536z^{14} + 148972z^{15} + 141835z^{16} + 85044z^{17} \\ &\quad + 58406z^{18} + 17822z^{19} + 7429z^{20}. \end{aligned} \tag{5.46}$$

Rosengren [35] also proved that there are polynomials $p_n^{(R)}$ (not to be confused with the polynomials p_n of Bazhanov and Mangazeev) with integer coefficients, such that the partition function of the three-color model with DWBC can be written in terms of these. These polynomials are conjectured to have positive coefficients. In [37], Rosengren shows that these polynomials can be written in terms of the functions $t^{(k_0,k_1,k_2,k_3)}(\psi)$ as

$$p_n^{(R)}(\psi) = \frac{(-1)^{\lfloor n/2 \rfloor}(\psi/2+1)^{n(n-1)-\lfloor (n-1)^2/4 \rfloor}}{(1-\psi)\psi^{n(n-1)}(\psi+1)^{n^2-2n-1}(2\psi+1)^{\lfloor n^2/4 \rfloor}} t^{(n+1,n,0,-1)}(\psi). \tag{5.47}$$

Since T is equivalent to Filali's determinant, all polynomials s_n, $\bar{s}_n$, p_n, q_n and $p_N^{(R)}$ are special cases of the 8VSOS partition function with DWBC and a reflecting end. This indicates that the three-color model with DWBC and a reflecting end could be used to interpret some of these polynomials combinatorially. This is the objective of Paper I and Paper II.

6 Summary of papers

6.1 Paper I - A combinatorial description of certain polynomials related to the XYZ spin chain

The goal of Paper I is to study the connection between the three-color model with DWBC and a reflecting end and the polynomials q_n introduced by Bazhanov and Mangazeev (see Section 5.1). Following Kuperberg [22] and Rosengren [34], we specify the parameters $\eta = -2/3$, $\lambda_i = -1/2$ and $\mu_j = 0$ in the partition function of the 8VSOS model with DWBC and a reflecting end (see Section 4). This specialization of the 8VSOS partition function can be written in terms of the partition function for the three-color model with the same boundary conditions (see Section 2.3) as

$$\begin{aligned} &Z_n(\underbrace{\omega,\dots,\omega}_{n},\underbrace{1,\dots,1}_{n},\rho,\zeta) \\ &= (-1)^{\binom{n}{2}} \sum_{m=0}^{n} \left(\frac{\vartheta(\rho\zeta\omega^{-1})}{\vartheta(\rho\zeta\omega)}\right)^{m} \left(\frac{\vartheta(\zeta\omega^{-1})}{\vartheta(\zeta\omega)}\right)^{n-m} \omega^{n^2+n-m} \\ &\quad \times \vartheta(\rho^3,p^3)^{2n^2+2n}\vartheta(\rho)^{n+3}\vartheta(\rho\omega^{-1})^{2m}\vartheta(\rho\omega)^{2(n-m)}B \\ &\quad \times Z^{3C}_{n,m}\left(\frac{1}{\vartheta(\rho)^3},\frac{1}{\vartheta(\rho\omega)^3},\frac{1}{\vartheta(\rho\omega^2)^3}\right), \end{aligned} \tag{6.1}$$

with

$$B = \begin{cases} 1, & \text{for } n \equiv 0 \mod 3, \\ \dfrac{\vartheta(\rho\omega^{-1})}{\vartheta(\rho)}, & \text{for } n \equiv 1 \mod 3, \\ \dfrac{\vartheta(\rho\omega)\vartheta(\rho\omega^{-1})}{\vartheta(\rho)^2}, & \text{for } n \equiv 2 \mod 3, \end{cases} \tag{6.2}$$

and where

$$Z^{3C}_{n,m}(t_0,t_1,t_2) = \sum_{\substack{\text{states with}\\ m \text{ turns of color 2}}} \prod_{\text{faces}} t_a \tag{6.3}$$

is the partition function of all three-colorings with a given number m of turns with color 2, and where t_a is the weight assigned to color a.

We then rewrite Filali's determinant in terms of Rosengren's functions T, which we can then express in terms of Bazhanov's and Mangazeev's polynomials q_n by using (5.39). Thus we get another expression for the partition function in terms of q_n, namely,

$$\begin{aligned} &Z_n(\omega,\dots,\omega,1,\dots,1,\rho,\zeta) \\ &= (-1)^{\binom{n}{2}} \sum_{m=0}^{n} \left(\frac{\vartheta(\rho\zeta\omega^2)}{\vartheta(\rho\zeta\omega)}\right)^m \left(\frac{\vartheta(\zeta\omega^{-1})}{\vartheta(\zeta\omega)}\right)^{n-m} \omega^{n^2+n-m} \\ &\quad\times \binom{n}{m} \frac{\vartheta(\rho\omega)^m \vartheta(\rho\omega^{-1})^{n-m}}{\vartheta(\rho)^n} \left(\frac{\vartheta(-\omega)^2\vartheta(-p^{1/2})\vartheta(p^{1/2}\omega)^3}{\vartheta(\omega)^2\vartheta(-1)\vartheta(p^{1/2})^2\vartheta(-p^{1/2}\omega)^4}\right)^{n(n-1)} \\ &\quad\times \tilde{B}\left((\psi+1)(2\psi+1)^2\right)^{n(n-1)} q_{n-1}\left(\frac{1}{2\psi+1}\right), \end{aligned} \tag{6.4}$$

where

$$\tilde{B} = \begin{cases} 1, & \text{for } n \equiv 0,2 \mod 3, \\ \dfrac{\vartheta(\rho\omega^{-1})}{\vartheta(\rho)}, & \text{for } n \equiv 1 \mod 3, \end{cases} \tag{6.5}$$

and where ψ is given by (5.20). Combining the two expressions (6.1) and (6.4), we get the partition function of the three-color model in terms of the polynomials q_n. By changing variables to $t_a = 1/\vartheta(\rho\omega^a)^3$ and $z = 1/(2\psi+1)$, we reach our main result.

Theorem 6.1 (Theorem 1.1 in Paper I). *Let t_a be the weight assigned to color a, and let m be the number of faces on the left boundary with color 2. For any m, it holds that*

$$\begin{aligned} &Z^{3C}_{n,m}(t_0,t_1,t_2) \\ &= \begin{cases} \binom{n}{m}\frac{t_2^{m-n}}{t_1^m}\frac{t_0(t_0t_1t_2)^{(2n^2+4n)/3}}{(z(z^2-1))^{(n^2-n)/3}} q_{n-1}(z), & n \equiv 0,1 \mod 3, \\ \binom{n}{m}\frac{t_2^{m-n}}{t_1^m}\frac{(t_0t_1+t_0t_2+t_1t_2)(t_0t_1t_2)^{(2n^2+4n-1)/3}}{(3z^2+1)(z(z^2-1))^{(n^2-n-2)/3}} q_{n-1}(z), & n \equiv 2 \mod 3, \end{cases} \end{aligned} \tag{6.6}$$

where

$$\frac{(t_0t_1+t_0t_2+t_1t_2)^3}{(t_0t_1t_2)^2} = \frac{(3z^2+1)^3}{(z(z^2-1))^2}. \tag{6.7}$$

As we saw in Section 5.4, $q_n(z)$ can be computed recursively (5.40). By inserting the expressions for $q_n(z)$ into Theorem 6.1, and simplifying with (6.7), we can compute the partition function for the three-color model with reflecting end and DWBC. Here we have computed $Z^{3C}_{n,m}(t_0, t_1, t_2)$ for some small n and m,

$$\begin{aligned}
Z^{3C}_{1,0}(t_0,t_1,t_2) &= t_0^3 t_1^2 t_2, \\
Z^{3C}_{1,1}(t_0,t_1,t_2) &= t_0^3 t_1 t_2^2, \\
Z^{3C}_{2,0}(t_0,t_1,t_2) &= t_0^6 t_1^6 t_2^3 + t_0^6 t_1^5 t_2^4 + t_0^5 t_1^6 t_2^4, \\
Z^{3C}_{2,1}(t_0,t_1,t_2) &= 2(t_0^6 t_1^5 t_2^4 + t_0^6 t_1^4 t_2^5 + t_0^5 t_1^5 t_2^5), \\
Z^{3C}_{2,2}(t_0,t_1,t_2) &= t_0^6 t_1^4 t_2^5 + t_0^6 t_1^3 t_2^6 + t_0^5 t_1^4 t_2^6, \\
Z^{3C}_{3,0}(t_0,t_1,t_2) &= t_0^9 t_1^8 t_2^5 (t_0 t_1 + t_0 t_2 + t_1 t_2)^3 - t_0^{11} t_1^{10} t_2^7, \\
Z^{3C}_{4,0}(t_0,t_1,t_2) &= t_0^{13} t_1^{12} t_2^8 (t_0 t_1 + t_0 t_2 + t_1 t_2)^6 - 3 t_0^{15} t_1^{14} t_2^{10} (t_0 t_1 + t_0 t_2 + t_1 t_2)^3 \\
&\quad - 2 t_0^{17} t_1^{16} t_2^{12}.
\end{aligned} \tag{6.8}$$

Inserting $t_0 = t_1 = t_2 = 1$ yields the number of three-colorings (see further Corollary 6.5 in Paper I).

Let $N^{(m)}(k_0, k_1, k_2)$ denote the number of states in the three-color model with exactly m positive turns and k_i faces of color i, then

$$Z^{3C}_{n,m}(t_0,t_1,t_2) = \sum_{(k_0,k_1,k_2)\in\mathbb{Z}^3} N^{(m)}(k_0,k_1,k_2) t_0^{k_0} t_1^{k_1} t_2^{k_2}. \tag{6.9}$$

Similarly, let $N^{(0)}(k_0)$ denote the number of states with m positive turns and k_0 faces of color 0. By inserting $t_0 = z(z+1)/(z-1)^2$, $t_1 = t_2 = 1$ and $m = 0$ into Theorem 6.1, we get

$$q_{n-1}(z) = \sum_{k_0\in\mathbb{Z}} N^{(0)}(k_0)(z(z+1))^{k_0-(n^2+5n+a)/3}(z-1)^{(5n^2+7n+2a)/3-2k_0}, \tag{6.10}$$

with $a = 3$, for $n \equiv 0, 1 \bmod 3$, and $a = 1$, for $n \equiv 2 \bmod 3$, and

It is clear that the coefficients of $q_n(z)$ are integers. Unfortunately the above does not prove the conjecture that $q_n(z)$ has only positive integer coefficients. It is not enough to notice that $N^{(0)}(k_0)$ is always non-negative, one would need some further constraints on $N^{(0)}(k_0)$. However, we have a weaker result.

Corollary 6.2 (Corollary 6.1 in Paper I)**.** *The polynomials $(z+1)^{n(n+1)} q_n\left(\frac{1}{z+1}\right)$ and $q_n(z+1)$ have positive integer coefficients.*

We also get some non-trivial results for the three-color model.

Corollary 6.3 (Corollary 6.2 in Paper I). *Let $N^{(m)}(k_0, k_1, k_2)$ be the number of states with m positive turns and k_i faces of color i. Then*

$$N^{(m)}(k_0, k_1, k_2) = \binom{n}{m} N^{(0)}(k_0, k_1 + m, k_2 - m). \tag{6.11}$$

Corollary 6.4 (Corollary 6.3 in Paper I). *Let $N^{(m)}(k_0, k_1, k_2)$ be the number of states with m positive turns and k_i faces of color i. Then*

$$N^{(m)}(k_0 + d, k_1 - m, k_2 + m - n), \tag{6.12}$$

with

$$d = \begin{cases} 1, & n \equiv 0, 1 \mod 3, \\ 0, & n \equiv 2 \mod 3, \end{cases} \tag{6.13}$$

is a symmetric function of k_0, k_1, k_2.

Furthermore we discuss the states with the maximum and minimum number of faces of each specific color. We have the following corollary. Some parts of the corollary are easy to prove combinatorially, while others seem harder to understand, see the discussion after the proof in Paper I.

Corollary 6.5 (Corollary 6.4 in Paper I). *Let $N_i^{(m)}(k)$ be the number of states with m positive turns and k faces of color i. For each m, the number of states with the minimum number of faces of each color is*

$$\begin{aligned} N_0^{(m)}\left(\frac{n^2 + 5n + a}{3}\right) &= N_1^{(m)}\left(\frac{n^2 + 5n + c}{3} - m\right) \\ &= N_2^{(m)}\left(\frac{n^2 + 2n + c}{3} + m\right) = \binom{n}{m}, \end{aligned} \tag{6.14}$$

and the number of states with the maximum number of faces of each color is

$$\begin{aligned} N_0^{(m)}\left(\frac{5n^2 + 7n + 2a}{6}\right) &= N_1^{(m)}\left(\frac{5n^2 + 7n + 2c}{6} - m\right) \\ &= N_2^{(m)}\left(\frac{5n^2 + n + 2c}{6} + m\right) = \binom{n}{m} 2^{n(n-1)/2}, \end{aligned} \tag{6.15}$$

where

$$a = \begin{cases} 3, & \text{for } n \equiv 0, 1 \mod 3, \\ 1, & \text{for } n \equiv 2 \mod 3, \end{cases} \quad \text{and} \quad c = \begin{cases} 0, & \text{for } n \equiv 0, 1 \mod 3, \\ 1, & \text{for } n \equiv 2 \mod 3. \end{cases} \tag{6.16}$$

6.2 Paper II - A combinatorial description of certain polynomials related to the XYZ spin chain. II. The polynomials p_n

Paper II is a continuation of Paper I. In Paper I, we studied the polynomials q_n, and in Paper II, we perform a similar study of the polynomials p_n of Bazhanov and Mangazeev (see Section 5.1). We specify all but one of the parameters in the partition function of the 8VSOS model with DWBC and a reflecting end in Kuperberg's way. We specialize[1] $\eta = -2/3$, $\lambda_i = 1$ and $\mu_j = 0$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$. The last parameter we specialize as $\mu_n = 1/4$ to find a connection to the polynomials p_n. The main idea is the same as in Paper I, but due to the different specialization of one of the parameters, the expressions we encounter are more involved than those in Paper I, and sometimes we need to go about things in a different way. We get a connection to a special case of the three-color model.

We rewrite Filali's determinant formula in terms of Rosengren's functions T, which we can then express in terms of Bazhanov's and Mangazeev's polynomials p_n by using (5.38). The partition function is then

$$
\begin{aligned}
&Z_n(\underbrace{\omega,\dots,\omega}_{n},\underbrace{1,\dots,1}_{n-1},-\omega,\rho,\zeta)\\
&= (-\omega)^{\binom{n+1}{2}+n}\left(\frac{\vartheta(-\omega)^2\vartheta(-p^{1/2})}{\omega\vartheta(-1)\vartheta(p^{1/2})^2\vartheta(p^{1/2}\omega)\vartheta(-p^{1/2}\omega)^4\vartheta(\omega)^2}\right)^{n^2-n}\\
&\quad\times\frac{\tilde{B}\,\vartheta(p^{1/2}\omega)^{2(n-1)(2n-1)}(\vartheta(-p^{1/2}\omega^2)\vartheta(-p^{1/2}))^{n-1}}{\vartheta(\rho)^n\vartheta(\omega)}\\
&\quad\times\left(\frac{\psi}{2\psi+1}\right)^{n-1}((\psi+1)(2\psi+1)^2)^{n^2-n}p_{n-1}\left(-\frac{1}{2\psi+1}\right)\\
&\quad\times\sum_{m=0}^{n}\left(\binom{n-1}{m-1}\omega^{-m}\vartheta(-1)\vartheta(-\rho\omega^2)\vartheta(\rho\omega)^{m-1}\vartheta(\rho\omega^2)^{n-m}\right.\\
&\quad\left.-\binom{n-1}{m}\omega^{-m-2}\vartheta(-\omega)\vartheta(-\rho)\vartheta(\rho\omega)^m\vartheta(\rho\omega^2)^{n-m-1}\right)\\
&\quad\times\left(\frac{\vartheta(\rho\zeta\omega^2)}{\vartheta(\rho\zeta\omega)}\right)^m\left(\frac{\vartheta(\zeta\omega^2)}{\vartheta(\zeta\omega)}\right)^{n-m},
\end{aligned}
\tag{6.17}
$$

[1]Remark: The specialization $\lambda_i = 1$ yields the same specialization of the parameters in the partition function as the specialization $\lambda_i = -1/2$ in Paper I, since $q^{-1/2} = q = \omega$.

where

$$\tilde{B} = \begin{cases} 1, & \text{for } n \equiv 0,2 \mod 3, \\ \dfrac{\vartheta(\rho\omega^{-1})}{\vartheta(\rho)}, & \text{for } n \equiv 1 \mod 3. \end{cases} \tag{6.18}$$

The partition function can also be written in terms of the partition function of the three-color model. The fact that one of the spectral parameters is specified differently than the others has the consequence that we need to split the partition function of the 8VSOS model into a sum of partition functions of special states of the three-color model, where we, besides specifying the colors on the boundaries, also specify the colors in the column closest to the right boundary (see Figure 6.1). The partition function for the 8VSOS model can then be written as a sum of all such specialized partition functions, as

$$\begin{aligned} &Z_n(\omega,\dots,\omega,1,\dots,1,-\omega,\rho,\zeta) \\ &= (-1)^{\binom{n+1}{2}}\omega^{-\binom{n+1}{2}+1}\frac{\vartheta(\rho)^{n+3}\vartheta(-1)^{2n-1}\vartheta(\rho^3,p^3)^{2n(n+1)-1}}{\vartheta(\omega)^{2n-1}}X \\ &\times \sum_{m=0}^{n}\frac{\vartheta(\rho\omega)^{2(n-m)}\vartheta(\rho\omega^2)^{2m}}{\omega^{2(n-m)}}\left(\frac{\vartheta(\rho\zeta\omega^2)}{\vartheta(\rho\zeta\omega)}\right)^m\left(\frac{\vartheta(\zeta\omega^2)}{\vartheta(\zeta\omega)}\right)^{n-m} \\ &\times \sum_{l=0}^{2n}(-1)^l\omega^l\left(\frac{\vartheta(-\omega)}{\vartheta(-1)}\right)^{l-1}\vartheta(\rho\omega^{-n+l})\vartheta(\rho\omega^{-n+l-1})\vartheta(-\rho\omega^{-n+l+2}) \\ &\times \sum_{\substack{\text{states with} \\ m \text{ positive turns} \\ \text{and} \leftarrow \text{on } l\text{th row}}}\prod_{\text{faces}}\frac{1}{\vartheta(\rho\omega^a)^3}, \end{aligned} \tag{6.19}$$

for

$$X = \begin{cases} 1, & n \equiv 0 \mod 3, \\ \dfrac{\vartheta(\rho\omega^2)}{\vartheta(\rho)}, & n \equiv 1 \mod 3, \\ \dfrac{\vartheta(\rho\omega)\vartheta(\rho\omega^2)}{\vartheta(\rho)^2}, & n \equiv 2 \mod 3, \end{cases} \tag{6.20}$$

and where a is the height of each face respectively. Here we choose to express the partition function in terms of arrows on the edges, but because of the bijection to three-colorings, one could equally well think of it as a result for the three-colorings. That a state has its left pointing arrow on the lth row from below means for the three-coloring that when starting from the bottom of the second to last column to the right, the colors $0, 1$ and 2 change in ascending order modulo 3, except when it crosses the lth edge, where it decreases by $1 \bmod 3$, to then continue the ascending order, see Figure 6.1.

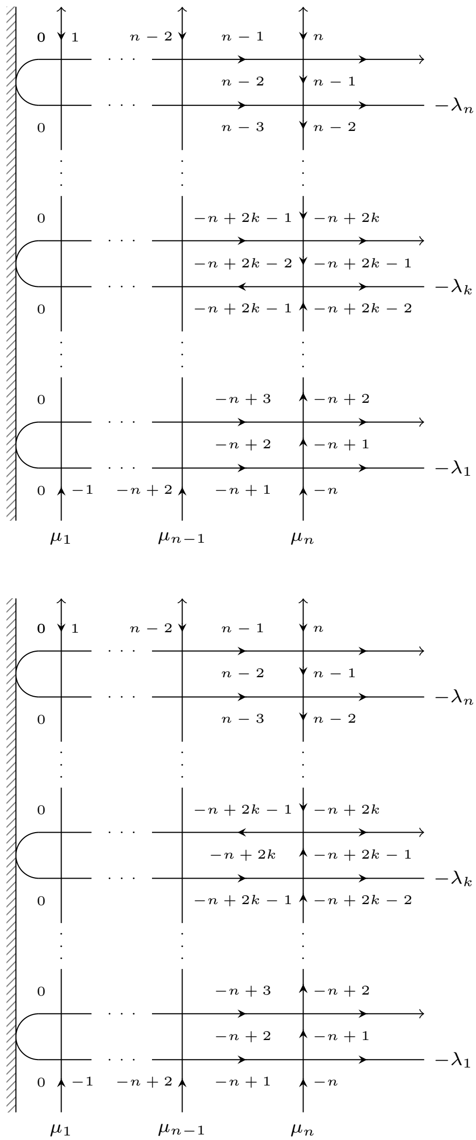


Figure 6.1: On the edges between the $(n-1)$th and nth column of vertices, there is exactly one left arrow in each state. The single left arrow is on row $l = 2k-1$ counted from below in the first lattice, and on row $l = 2k$ counted from below in the second lattice.

Due to the summation over l, it seems hard to study the full partition function of the three-color model as we did in Paper I. Instead, we consider the special cases $\rho = -\omega^a$, $a = 0, 1, 2$. This allows us to study cases of the three-color model where two colors have equal weight. We identify the terms with the same m in the two expressions (6.17) and (6.19) and then we change to the variable $z = -1/(2\psi + 1)$. Then we get the following theorem.

Theorem 6.6 (Theorem 6.1 in Paper II)**.** *Formulas for $p_{n-1}(z)$ are given by*

$$\binom{n-1}{m-1} p_{n-1}(z) = \sum_{\substack{l \equiv n \bmod 3 \\ k_0 \in \mathbb{Z}}} \left((-1)^{n+l} \left(N_{m,l}(k_0) + N_{m,l-1}(k_0) \right) \times \frac{(z(z-1))^{(3k_0 - n^2 - 6n + l - c)/3}}{(z+1)^{(6k_0 - 5n^2 - 9n + 2l - 2c)/3}} \right), \tag{6.21}$$

$$\binom{n}{m} p_{n-1}(z) = \sum_{\substack{l \equiv n \bmod 3 \\ k_1 \in \mathbb{Z}}} \left((-1)^{n+l} \left(N_{m,l}(k_1) + N_{m,l+1}(k_1) \right) \times \frac{(z(z-1))^{(3k_1 - n^2 - 6n + 3m + l - d)/3}}{(z+1)^{(6k_1 - 5n^2 - 9n + 6m + 2l - 2d)/3}} \right), \tag{6.22}$$

and

$$\binom{n-1}{m} p_{n-1}(z) = \sum_{\substack{l \equiv n \bmod 3 \\ k_2 \in \mathbb{Z}}} \left((-1)^{n+l} \left(N_{m,l+1}(k_2) + N_{m,l+2}(k_2) \right) \times \frac{(z(z-1))^{(3k_2 - n^2 - 3n - 3m + l - d)/3}}{(z+1)^{(6k_2 - 5n^2 - 3n - 6m + 2l - 2d)/3}} \right), \tag{6.23}$$

where

$$c = \begin{cases} 3, & n \equiv 0, 1 \mod 3, \\ 1, & n \equiv 2 \mod 3, \end{cases} \qquad \text{and} \qquad d = \begin{cases} 0, & n \equiv 0, 1 \mod 3, \\ 1, & n \equiv 2 \mod 3, \end{cases} \tag{6.24}$$

and $N_{m,l}(k_i)$ is the number of states with m positive turns, k_i faces of color i and where the left arrow of the second to last column is on the lth row from below.

Although this gives combinatorial expressions for the polynomials p_n, we can not see from them that the coefficients of p_n are positive.

6.3 Paper III - Exact results for the six-vertex model with domain wall boundary conditions and a partially reflecting end

In Paper III, we consider the trigonometric 6V model on a lattice of size $2n \times m$, $m \leq n$, with DWBC and a partially reflecting end, i.e. with a triangular K-matrix, extending the work of Foda and Zarembo [17] to the trigonometric case. We obtain a determinant formula for the partition function by using the standard method of Izergin and Korepin, see Section 3.3. In an appendix, we instead use another method, by Foda and Wheeler, to find the determinant formula for the partition function.

We parametrize the local weights slightly differently in Paper III than in Section 3. We define $f(x) = e^x - e^{-x}$, and define the weights as[2]

$$\begin{aligned} &a_\pm(\lambda) = 1, \qquad b_\pm(\lambda) = e^{\mp\eta}\frac{f(\lambda)}{f(\lambda+\eta)}, \qquad c_\pm(\lambda) = e^{\pm\lambda}\frac{f(\eta)}{f(\lambda+\eta)}, \\ &k_\pm(\lambda,\zeta) = e^{\zeta\mp\lambda}f(\zeta\pm\lambda), \qquad k_c(\lambda,\zeta) = \varphi f(2\lambda). \end{aligned} \tag{6.25}$$

The first main theorem is the following.

Theorem 6.7 (Theorem 3.5 in Paper III)**.** *For the 6V model with DWBC and a partially reflecting end on a lattice of size $2n \times m$, $m \leq n$, the partition function is*

$$\begin{aligned} Z_{n,m}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \varphi^{n-m} e^{(\binom{m}{2}-nm)\eta} f(\eta)^m \prod_{i=1}^{m}\left[e^{\mu_i+\zeta}f(\mu_i-\zeta)\right]\prod_{i=1}^{n} f(2\lambda_i) \\ &\times \frac{\prod_{i=1}^{n}\prod_{j=1}^{m} f(\mu_j\pm\lambda_i)}{\prod_{1\leq i<j\leq m} f(\mu_j\pm\mu_i)\prod_{1\leq i<j\leq n}\left[f(\lambda_i-\lambda_j)f(\lambda_i+\lambda_j+\eta)\right]} \det_{1\leq i,j\leq n} M, \end{aligned} \tag{6.26}$$

where M is an $n \times n$ matrix with

$$M_{ij} = \begin{cases} \dfrac{1}{f(\mu_i\pm\lambda_j)f(\mu_i\pm(\lambda_j+\eta))}, & \text{for } i \leq m, \\ h((n-i)(2\lambda_j+\eta)), & \text{for } m < i < n, \\ 1, & \text{for } i = n, \end{cases} \tag{6.27}$$

where $f(x) = 2\sinh x$, $h(x) = 2\cosh x$ and where $f(x \pm y) = f(x+y)f(x-y)$.

[2]In Paper III, we use the letter γ instead of η.

Thereafter we specify the spectral parameters in Kuperberg's manner in the definition of the partition function as well as in the determinant formula to finally end up with an expression for the number of states. It is not a priori clear that one can do so. In the case of partial DWBC on an $n \times m$ lattice, it seems hard since phases appear [16]. However, in the case of partial reflection and DWBC, similar phases do not appear, due to the alternating orientations of the lines.

After specializing the spectral parameters in the determinant formula, we show that the determinant has a Hankel determinant part and a Vandermonde determinant part. Following ideas of Colomo and Pronko [11], we show that the determinant can be written in terms of certain orthogonal polynomials, namely, the *Wilson polynomials*

$$W_k\left(\left(\frac{t}{6}\right)^2; \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right) = (5/6)_k(4/3)_k k! \sum_{j=0}^{k} \frac{(-k)_j(3/2+k)_j(1/3+\mathrm{i}t/6)_j(1/3-\mathrm{i}t/6)_j}{(5/6)_j(4/3)_j(j!)^2}, \tag{6.28}$$

where $(a)_j = a(a+1)\cdots(a+j-1)$, for $j \geq 1$, and $(a)_0 = 1$ is the rising factorial. This yields our second main result.

Theorem 6.8 (Theorem 5.1 in Paper III)**.** *For the 6V model with DWBC and a partially reflecting end, the number of states with exactly k turns of type k_+ is*

$$N_k = \binom{m}{k} \frac{2^{n^2-n-m^2-m}(n-m)!}{3^{2m^2-m-n^2+n-mn}} \prod_{j=1}^{n} \frac{(2j-2)!}{(4j-3)!} \prod_{j=1}^{m} \frac{(6j-2)!}{(4j-1)!} \prod_{j=m+1}^{n} \frac{1}{(j-1)!} \times \det_{1\leq l,j\leq n-m}\left(W_{m+j-1}\left(-\frac{l^2}{9}; \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right)\right). \tag{6.29}$$

By summing over k, we get that the total number of states of the model is

$$A(m,n) \coloneqq \sum_{k=0}^{m} N_k = \frac{2^{n^2-n-m^2}(n-m)!}{3^{2m^2-m-n^2+n-mn}} \prod_{j=1}^{n} \frac{(2j-2)!}{(4j-3)!} \prod_{j=1}^{m} \frac{(6j-2)!}{(4j-1)!} \times \prod_{j=m+1}^{n} \frac{1}{(j-1)!} \det_{1\leq l,j\leq n-m}\left(W_{m+j-1}\left(-\frac{l^2}{9}; \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right)\right). \tag{6.30}$$

We also show that we can rewrite N_k as an $(n-m)$-fold hypergeometric sum,

$$
\begin{aligned}
N_k = & \binom{m}{k} \frac{2^{n^2-n-m^2-m}(n-m)!}{3^{2m^2-m-n^2+n-mn}} \prod_{j=1}^{n} \frac{(2j-2)!}{(4j-3)!} \prod_{j=1}^{m} \frac{(6j-2)!}{(4j-1)!} \\
& \times \prod_{j=m+1}^{n} (5/6)_{j-1}(4/3)_{j-1} \prod_{j=1}^{n-m} \frac{(5/2+2n-2j)_{j-1}}{(1-n)_{j-1}(m+3/2)_{j-1}} \\
& \times \sum_{l_1,l_2,\ldots,l_{n-m}=0}^{n-1} \left(\prod_{i=1}^{n-m} \frac{((1-i)/3)_{l_i}((1+i)/3)_{l_i}(1-n)_{l_i}(m+3/2)_{l_i}}{(5/6)_{l_i}(4/3)_{l_i}(l_i!)^2} \right. \\
& \left. \times \prod_{1\le i<j\le n-m} (l_i - l_j) \right). \qquad (6.31)
\end{aligned}
$$

Furthermore we show that there is a bijection from the states of the 6V model to a type of ASM-like objects described in Section 3.5.

Proposition 6.9 (Proposition 4.3 in Paper III)**.** *Consider matrices of size $2n \times m$, $m \le n$, consisting of elements 0, -1 and 1, for which the following properties hold. Vertically and horizontally the nonzero elements alternate in sign and the sum of the elements of each column is 1. Horizontally the rows are pairwise connected to the nearest row at the left edge to form a double row. If a row has any nonzero elements, the rightmost of these must be 1. Furthermore the sum of the entries in a double row must be 0 or 1. Then the expression $A(m,n)$ yields the number of such matrices.*

It also follows that N_k counts the number of matrices equivalent to states of the 6V model with k positive turns. In the case $m = n$, the sum of the elements in each double row must be 1, and $A(m,n)$ counts the number of UASMs [23]. The case where $m = n$ and $k = 0$ corresponds to VSASMs.

7 Future problems

In Paper I, we restricted ourselves to the case $\lambda_i = 1$, $\mu_j = 0$. In this case, the partition function of the 8VSOS model with DWBC and a reflecting end is a factor times $t^{(2n,0,0,0)}$, which is connected to the polynomials q_{n-1} by (5.39). In Paper II, we specialize $\lambda_i = 1$, $\mu_j = 0$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, and $\mu_n = 1/4$. This lets us write the partition function in terms of $t^{(2n+1,0,0,-1)}$, which is connected to p_n by (5.38). Another natural case to investigate is when $\lambda_i = 1$, $\mu_j = 0$, for $1 \leq i \leq n-1$ and $1 \leq j \leq n$, and where λ_n is specialized differently.

One could of course also consider other choices of the parameters λ_i and μ_j, for instance to get $t^{(n,n,0,0)}$ which is connected to s_n by (5.36). It would also be possible to keep one parameter free, e.g. λ_k, and look at $T_n^{(2n-1,0,0,0)}(\lambda_k)$. For the 6V model, one free parameter corresponds to refined enumerations of ASMs, which counts the number of ASMs where the unique 1 of the first row is in a given column [43]. Refined enumerations of UASMs have been studied e.g. by [31]. Following [34], another idea would be to compute the probabilities that a random face from a random three-coloring has color i.

In Paper III, we consider the 6V model of size $2n \times m$, where $m \leq n$, with DWBC and a partially reflecting end. We find the partition function and can then count the number of states. Here the reflection matrix is an upper triangular matrix. It would be interesting to perform a similar investigation with a lower triangular reflection matrix, corresponding to the case where $m \geq n$. Another natural research idea is to explore the case with a full reflection matrix. In both these cases it is not clear how to find recursion formulas for the Izergin–Korepin procedure. The frozen regions in the proofs of Tsuchiya [41], Foda and Zarembo [17] and the proof in Paper III are obtained since a particular element in the K-matrix is zero.

The model in Paper III is connected to a type of generalized UASMs. Kuperberg [23] presented several other types of ASMs and vertex models, with different boundary conditions. For instance there are the double U-turn ASMs (UUASMs), with U-turns on the left side and also on the top, and several types of models with a square lattice on a triangular domain, e.g. OSASMs, OOSASMs, UOSASMs. Other future problems would be to try to find determinant or Pfaffian formulas for the partition functions of the 8VSOS generalizations of these models. The next step would be to look at the corresponding three-colorings.

Pozsgay [28] used the homogeneous limit of Tsuchiya's $2n \times n$ determinant to compute overlaps (i.e. inner products) between (off-shell) Bethe states of the XXZ spin chain and certain simple product states, such as the Néel states. In a similar way, Foda and Zarembo [17] used their rational $2n \times m$ determinant formula to compute the overlaps between Bethe states of the XXX spin chain and more general objects which they call partial Néel states. This suggests that it could be possible to compute similar overlaps in the trigonometric case of the XXZ spin chain.

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