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Abstract

We study whether rational information processing is testable. Our main result shows that, under positive conditions, negative introspection holds if and only if it holds for primitive propositions. In particular, it is sufficient to test negative introspection on primitive propositions.

Keywords: Partitional possibility correspondence, negative introspection, primitive propositions.

JEL Classification: D80, D83, D89.

1. Introduction

One of the central components of modern economics is the consideration that agents are limited in their means to access and treat information. The "bounded rationality" literature, which specifically addresses this issue, dates back to Simon [20], and has surged in the last decades (Aumann [2]; Lipman [13]). Rationality and its lack thereof can be observed both in the agents' decisions, and their information processing. Failures of rationality in the information processing are more fundamental than in decision taking, since non-rational information processing entails non-rational decision making, whilst the converse is not necessarily true.

Hintikka [11], Aumann [1], and Geanakoplos [7] introduced a semantic model of information structures that represent the information processing of both perfectly, and non-perfectly rational agents respectively (see also Brandenburger, et al. [5]; and Dekel, et al. [6]). It is commonly argued that the possibility correspondence of a perfectly rational agent has to be partitional, since the agent can exclude all the states with different information, and no others. Hence, rational agents should exhibit partitional possibility correspondences, and non-partitional possibility correspondences should be taken as a sign of irrationality (see, e.g, chapter 3 of Rubinstein [17]).

An important question is whether one can test whether an agent's possibility correspondences is partitional or not. Recall that, by definition, an agent's information is partitional if, when the agent being in state ω considers the state ω' as possible, the same agent being at ω' would also consider ω as possible. Testing this is problematic, since it requires to observe the agent's knowledge in both states ω and ω'. Hence, the semantic model provides no direct way of testing whether an agent processes information rationally, or not.

Kripke [12], Bacharach [4], and Samet [18] introduced a syntactic model of knowledge in the form of a system of propositional calculus. This framework induces a larger state space, since knowledge, knowledge about knowledge, and so on, are explicitly embodied into the states. Information is then defined by the set of known propositions at every state.

A syntactic model of knowledge defines a semantic model in a natural way: the states the agent believes as possible are the states in which all the known propositions are true. A semantic model also defines a syntactic model: the known propositions are supersets of the set of states considered as possible. Nonetheless, because they allow for a variety of propositions and more elaborate state spaces, syntactic models are a more general framework than semantic ones.

In syntactic models, Bacharach [4], and Samet [18] independently showed that whenever the agent's knowledge satisfies the basic axioms of 1) knowledge, 2) positive introspection, and 3) negative introspection, the induced possibility correspondence is partitional. The axiom of knowledge says that if the agent knows some proposition, then this proposition is true. Under positive introspection, if the agent knows a proposition, then she also knows that she knows this proposition. Negative introspection says that if the agent does not know a proposition, she knows that she does not know it. Both the knowledge and positive introspection axioms are based on positive knowledge. On the other hand, negative introspection is based on knowledge oneself's ignorance.

We adopt the commonly accepted view that the knowledge and positive introspection axioms are rather non-problematic, and can be assumed if necessary. On the other hand, the negative introspection axiom is more controversial, and should be accepted or rejected based on empirical evidence only, and possibly also from logical implications derived from theoretical results.

Unlike partitional possibility correspondences in the semantic model, the axioms of knowledge, positive introspection and negative introspection are defined state by state. Therefore, observation of the agent's knowledge in particular states is enough to test whether these axioms are satisfied are not. Hence, syntactic models provide an appropriate framework in which the agent's information processing abilities can be tested.

The main difficulty arising from testing negative introspection is the infinite cardinality of the set of propositions. Indeed, every primitive proposition (fact) generates a sequence of epistemic (modal) propositions. We introduce an axiom of positive negation under which, if a proposition is known, it is known that its negation is not known. Positive negation, like knowledge and positive introspection, is based on positive knowledge by the agent, and is defined state by state. Our main result, Theorem 1, shows that under knowledge, positive introspection, and positive negation, negative introspection holds if and only it holds for the primitive propositions. Hence, negative introspection is testable, and is sufficient for partitional information structures.

Still, negative introspection is not necessary for partitional information structures. We provide an example of syntactic knowledge structure for which negative introspection is not satisfied and yet, the agent's induced possibility correspondence is partitional. This knowledge structure verifies the knowledge and positive introspection axioms, so that, even when those two properties can be assumed, a violation of negative introspection cannot alone be taken as evidence for a non partitional information structure.

We introduce an axiom of knowledge equivalence under which the definitions of knowledge arising from the syntactic model and from the corresponding semantic model coincide. Our Proposition 4 shows that, assuming knowledge equivalence, knowledge and positive introspection, negative introspection is equivalent to a partitional information structure. Although the equivalent of our Proposition 4 seems to have been known for semantic models (see Aumann [3]; Lipman [13]; Rubinstein [17]), we have not found a formal proof of this result in the literature.

We believe knowledge equivalence is a reasonable condition to assume. Indeed, Lemma 2 shows that this condition holds whenever the agent forms in the syntactic model the same logical implications she would form in the semantic model. The absence of equivalence of knowledge in the two models can be viewed as an artefact arising from the potentially high complexity of propositions in syntactic models.

Comparing the two conditions we introduce, we show in Lemma 3 that positive negation is weaker than knowledge equivalence. Therefore, under knowledge equivalence, testing partitional information structures is equivalent to testing negative introspection, which is in turn equivalent to testing negative introspection for the primitive propositions.

To summarize our results, observation of primitive knowledge is enough to test negative introspection under positive negation, and is enough to test partitional information structures under knowledge equivalence.

We recall the syntactic model of knowledge in Section 2. We show equivalence between negative introspection in the entire state space, and just the primitive propositions in Section 3. In Section 4, we show that under knowledge equivalence, negative introspection for the primitive propositions is, not only sufficient, but also necessary condition for a partitional possibility correspondence.

2. Knowledge

We recall the standard model of propositional calculus to model knowledge (Kripke [12]; Bacharach [4]; Samet, [18]). Let Φ be the countable set of propositions, which describe the relevant characteristics of the environment. The mappings $\kappa : \Phi \to \Phi$ and $\neg : \Phi \to \Phi$ stand for the propositions "the agent knows ϕ", and "not ϕ" respectively.

Let Ω_0 denote the set of all mappings $\omega : \Phi \to \{0, 1\}$ that satisfy complementarity: $\omega(\phi) = 1$ if and only if $\omega(\neg\phi) = 0$, for every $\omega \in \Omega_0$, and every $\phi \in \Phi$. We say that a proposition ϕ is true at a state $\omega \in \Omega_0$, and we write $\phi \in \omega$, if and only if $\omega(\phi) = 1$.

Alternatively a state ω can be identified by the set $\{\phi \in \Phi : \omega(\phi) = 1\}$, i.e., by the propositions that are true in this state. The ken (set of known propositions) at a state ω is defined as follows:

$$K(\omega) = \{\phi \in \Phi : \kappa\phi \in \omega\}. \tag{1}$$

Now consider a subset $\Omega \subset \Omega_0$. A state $\omega' \in \Omega$ is considered as possible while being at $\omega \in \Omega$ if every known proposition at ω is true at ω'. That is, the possibility correspondence $P : \Omega \to 2^\Omega$, maps every state ω to the set of states considered as possible by the agent while being at ω:

$$P(\omega) = \{\omega' \in \Omega : K(\omega) \subseteq \omega'\}. \tag{2}$$

A possibility correspondence is called *partitional* whenever $P(\omega) = P(\omega')$ for every $\omega' \in P(\omega)$, and every $\omega \in \Omega$.

The three fundamental properties of knowledge are:

(K_1) if $\kappa\phi \in \omega$ then $\phi \in \omega$ (axiom of knowledge),

(K_2) if $\kappa\phi \in \omega$ then $\kappa\kappa\phi \in \omega$ (positive introspection),

(K_3) if $\neg\kappa\phi \in \omega$ then $\kappa\neg\kappa\phi \in \omega$ (negative introspection).

Samet [18] defines the following state spaces:

Ω_1: the set of states that satisfy (K_1),

Ω_2: the set of states that satisfy $(K_1), (K_2)$,

Ω_3: the set of states that satisfy $(K_1), (K_2)$, and (K_3),

and proves the following result:

PROPOSITION 1. **(Samet [18])** *If $\Omega \subseteq \Omega_3$, the possibility correspondence P is partitional.*

The previous proposition follows directly from the fact that for $\Omega \subseteq \Omega_3$, $\omega' \in P(\omega)$ if and only if $K(\omega) = K(\omega')$, also proven by Samet [18]. In other words, $(K_1) - (K_3)$ imply that the possibility correspondences is such that the states considered as possible at ω are those that yield exactly the same knowledge as ω.

3. Primitive propositions and negative introspection

Proposition 1 provides a way to test for partitional information structures, since $K1-K3$ can be tested state by state. Still, if one wishes to do this, it is necessary to check (K_3) for every $\phi \in \Phi$ at every state ω. This would be practically impossible due to the infinite cardinality of Φ. Let $\Phi_0 \subset \Phi$ denote the (finite) set of primitive (atomic) propositions, which are not derived from some other proposition with the use of the knowledge operator κ. These propositions, which are also called non-epistemic or non-modal, refer to natural events (facts). Aumann [3] defines the primitive propositions as substantive happenings that are not described in terms of people knowing something. Bacharach [4], Samet [18], Modica and Rustichini [15], Hart, et al. [9], and Halpern [8] also discuss the distinction between primitive, and epistemic propositions.

Let $B_0(\phi) = \{\phi, \neg\phi\}$, and define inductively

$$B_n(\phi) = \{\kappa\phi', \neg\kappa\phi' | \phi' \in B_{n-1}(\phi)\}. \tag{3}$$

We call $B(\phi) = \bigcup_{n\geq 0} B_n(\phi)$ the set of propositions generated by ϕ. Obviously, the set of all propositions is given by the union of all primitive and epistemic propositions, i.e., $\Phi = \bigcup_{\phi\in\Phi_0} \bigcup_{n\geq 0} B_n(\phi)$. The cardinality of Φ is (countably) infinite, implying that testing whether (K_3) holds by observing every unknown proposition would be practically impossible. We propose an alternative way to figure out whether negative introspection holds or not, by only looking at the primitive propositions. Such a task would be easier, not only in terms of the cardinality of the propositions to be tested, but also in terms of complexity, since articulating high order epistemic propositions can be rather complicated.

The most disputed among the three basic axioms is negative introspection. Geanakoplos [7], and Lipman [13] note that it is often seen as a less realistic assumption, since people may not notice events that have not occurred. In general, whilst (K_1) and (K_2) are based on reasoning through knowledge, (K_3) assumes that the agent makes inference based on lack of knowledge. Instead of assuming (K_3), consider the following weaker axiom

(K_4) if $\kappa\phi \in \omega$ then $\kappa\neg\kappa\neg\phi \in \omega$ (positive negation),

which implies that if a proposition is known, then it is also known that its negation is not known. The axiom (K_4) relies on conclusions that can be drawn by the agent using positive knowledge (as opposed to (K_3)). It is easy to show that (K_4) is weaker than the following mild axiom (Modica and Rustichini [14]):

(K_I) if $\phi \in \omega$ implies $\phi' \in \omega$, then $\kappa\phi \in \omega$ also implies $\kappa\phi' \in \omega$ (axiom of inference).

Formally, if $\Omega \subseteq \Omega_2$ satisfies (K_I), then it also satisfies (K_4). This follows from the fact that given complementarity and (K_1), if $\kappa\phi \in \omega$, then $\neg\kappa\neg\phi \in \omega$. Then, it follows from (K_2) and (K_I) that $\kappa\neg\kappa\neg\phi \in \omega$, which implies (K_4). That is, if the agent is able to make simple deductions of the form of (K_I), then her knowledge satisfies (K_4). Since K_I is a reasonable axiom based on positive knowledge, it is not unrealistic to assume (K_4).

We define the state space

Ω_4: the set of states that satisfy $(K_1), (K_2)$, and (K_4).

It is straightforward that $\Omega_3 \subseteq \Omega_4$. This follows directly from the fact that, given complementarity and (K_1), if $\kappa\phi \in \omega$, then $\neg\kappa\neg\phi \in \omega$, and therefore from (K_4) follows directly from (K_3). However, the converse does not hold, i.e., (K_4) does not imply (K_3). Thus, $\Omega \subseteq \Omega_4$ isn't sufficient for the agent's knowledge to be partitional, as seen in the following example:

Example 1. Consider the state of complete ignorance ω_0 at which nothing is known: for every ϕ, $\neg\kappa\phi \in \omega_0$. Observe that $\omega_0 \in \Omega_4$ since K_1, K_2 and K_4 are based on positive knowledge (they require some implications whenever $\kappa\phi$ is true). Now consider any state $\omega_1 \in \Omega_4$ at which it is known that some ϕ is known ($\kappa\kappa\phi$ for some ϕ). Let Ω be any state space that contains ω_0 and ω_1. Since nothing is known at ω_0, $K(\omega_0) = \emptyset$, and $P(\omega_0) = \Omega$. On the other hand, $\kappa\phi \in K(\omega_1)$, so that $\omega_0 \notin P(\omega_1)$. Therefore, P is not partitional. $\triangleleft$

That is, some additional condition is required to ensure that the agent's knowledge is partitional. Assuming $\Omega \subseteq \Omega_4$, our main result below offers a practical way to test for partitional information, by observing the (finitely many, and easy to articulate) primitive propositions.

Theorem 1. *Consider* $\Omega \subseteq \Omega_4$. *Then* $\Omega \subseteq \Omega_3$ *if and only if* (K_3) *holds for every* $\phi \in \Phi_0$.

Theorem 1 shows that it suffices to look at whether negative introspection holds for the primitive, rather than all, propositions. Using this theorem, and the Bacharach-Samet result, we can corroborate that P is partitional just by fulfilling a much less demanding hypothesis that Proposition 1.

Corollary 1. *Consider* $\Omega \subseteq \Omega_4$, *and let* (K_3) *hold for every* $\phi \in \Phi_0$. *Then* P *is partitional.*

4. Primitive propositions in semantic models of knowledge

While knowledge in syntactic models is embodied to every state, in semantic models an event $E \subseteq \Omega$ is known at ω, and we write $\omega \in KE$, whenever $P(\omega) \subseteq E$ (Hinitkka [11]; Aumann [1]; Geanakoplos [7]). That is, an event is known whenever it occurs at every contingency considered as possible. For any proposition $\phi \in \Phi$ consider the following event:

$$E_\phi = \{\omega \in \Omega : \phi \in \omega\}. \tag{4}$$

The following result relates knowledge in syntactic and semantic models.

Proposition 2. *Consider* $\Omega \subseteq \Omega_0$. *If* $\kappa\phi \in \omega$, *then* $\omega \in KE_\phi$.

Proof. Let $\phi \in K(\omega)$. It follows from $K(\omega) \subseteq \omega'$ that $\phi \in \omega'$, for every $\omega' \in P(\omega)$. Hence $P(\omega) \subseteq E_\phi$. □

Note however that the converse of Proposition 2 does not hold in general. Consider for instance the following example:

Example 2. Consider the state space $\Omega \subseteq \Omega_4$:

$$\begin{aligned}
\omega_1 &= \{\phi, \kappa\phi, \neg\kappa\neg\phi, \kappa\kappa\phi, \kappa\neg\kappa\neg\phi, ...\}, \\
\omega_2 &= \{\neg\phi, \neg\kappa\phi, \kappa\neg\phi, \kappa\neg\kappa\phi, \kappa\kappa\neg\phi, ...\}, \\
\omega_3 &= \{\phi, \neg\kappa\phi, \neg\kappa\neg\phi, \neg\kappa\neg\kappa\phi, \neg\kappa\neg\kappa\neg\phi, \kappa\neg\kappa\neg\kappa\phi, \kappa\neg\kappa\neg\kappa\neg\phi, ...\}, \\
\omega_4 &= \{\neg\phi, \neg\kappa\phi, \neg\kappa\neg\phi, \neg\kappa\neg\kappa\phi, \neg\kappa\neg\kappa\neg\phi, \kappa\neg\kappa\neg\kappa\phi, \kappa\neg\kappa\neg\kappa\neg\phi, ...\}.
\end{aligned}$$

We have $P(\omega_3) = \{\omega_3, \omega_4\} \subseteq E_{\neg\kappa\phi} = \{\omega_2, \omega_3, \omega_4\}$, but $\neg\kappa\phi$ is not known at ω_3, i.e., $\neg\kappa\neg\kappa\phi \in \omega_3$. $\triangleleft$

Note that (K_3) is violated since at both ω_3 and ω_4 the agent does not know that she does not know ϕ, nor $\neg\phi$. That is, (K_3) is not required for a partitional possibility correspondence. If the converse to Proposition 2 is satisfied, then (K_3) is not only sufficient, but also a necessary condition for partitional information. This follows from the following result, which has been known for quite some time, but has never been explicitly stated and proven (see e.g. Aumann [3]; Lipman [13]; Rubinstein [17]).

Proposition 3. *Consider a state space Ω, and let P satisfy $\omega \in P(\omega)$, and $KE_\phi \subseteq KKE_\phi$. Then P is partitional if and only if $\setminus KE_\phi \subseteq K \setminus KE_\phi$.*

Then, by using Theorem 1 we can show that if the converse of Proposition 2 holds, the possibility correspondence is partitional if and only if negative introspection holds for primitive propositions. Formally, consider the axiom:

(K_E) If $\omega \in KE_\phi$, then $\kappa\phi \in \omega$ (knowledge equivalence).

In particular, under K_E, partitional information structures can be tested based on knowledge of primitive propositions.

Proposition 4. *Consider $\Omega \subseteq \Omega_2$, and let (K_E) hold in Ω. Then P is partitional if and only if (K_3) holds for $\phi \in \Phi_0$.*

The axiom of knowledge equivalence seems a quite reasonable assumption, since it only requires knowledge in the semantic and syntactic models to coincide. That is, if we believe that the semantic model appropriately describes the agent's knowledge, conclusions derived from the syntactic model should coincide with those of the semantic model.

Note finally that partitional knowledge is by nature a semantic property. As such, it requires observation of the entire state space in order to be verified or rejected. However, assuming KE_ϕ, partitional knowledge can be broken down to a state by state property, which can be tested by observing the knowledge at distinct states individually.

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Appendix

Lemma 1. *Consider $\Omega \subseteq \Omega_4$, and let $\phi \in K(\omega)$. Then $\phi' \in B(\phi)$ belongs to ω if and only if ϕ' contains an even number of negations.*

Proof. Every proposition in $B_n(\phi)$ contains n knowledge operators k and one proposition ϕ. The every $\phi' \in B_n(\phi)$ can be fully identified by a finite sequence of $n+1$ variables, where i-th variable takes value 1 if there is a negation in front of the i-th knowledge operator, and 0 otherwise. Therefore, $B_n(\phi)$ is in bijection with $\{0,1\}^{n+1}$, and hence has cardinality 2^{n+1}.

Let $B_n^e(\phi)$ denote the subset of $B_n(\phi)$ that contain an even number of negations. It follows from Newton's binomial applied for 1, and -1, that the cardinality of $B_n^e(\phi)$ is given by

$$\begin{aligned} C(B_n^e(\phi)) &= \sum_{\substack{k=2m \\ 2m\leq n+1}} \binom{n+1}{k} = 2^{n+1} - \sum_{\substack{k=2m+1 \\ 2m+1\leq n}} \binom{n}{k}, \\ C(B_n^e(\phi)) &= \sum_{\substack{k=2m \\ 2m\leq n+1}} \binom{n+1}{k} = 2^{n+1} + \sum_{\substack{k=2m+1 \\ 2m+1\leq n}} \binom{n}{k}. \end{aligned}$$

Summing the two equations, yields that the cardinality of $B_n^e(\phi)$ is equal to 2^n, which implies that exactly half of the propositions in $B_n(\phi)$ contain an even number of negations. It follows then by induction that if $\kappa\phi \in \omega$, then $B_n^e(\phi) \subseteq \omega$, which implies that $B^e(\phi) = \omega$, which completes the proof. $\square$

Proof of Theorem 1. It follows from Lemma 1 that all states with $\kappa\phi \in \omega$ contain all $\phi' \in B(\phi)$ with an even number of negations, and all states with $\kappa\neg\phi \in \omega'$ contain all $\phi' \in B(\phi)$ with an odd number of negations. In order for negative introspection to be violated, the state must contain states with both even and odd number of negations. When $\kappa\neg\kappa\phi \in \omega$ and $\kappa\neg\kappa\neg\phi \in \omega$, then we consider $\phi' = \neg\kappa\phi$, and $\phi'' = \neg\kappa\neg\phi$ separately, and from the previous argument (K_3) holds at ω, which proves the theorem. $\square$

Lemma 2. *Consider $\Omega \subseteq \Omega_0$ that satisfies (K_E). Then, for some $\phi \in \Phi$, and some $\omega \in \Omega$,*

(a) $KE_\phi \subseteq E_\phi$ if and only if (K_1),

(b) $KE_\phi \subseteq KKE_\phi$ if and only if (K_2),

(c) $\backslash KE_\phi \subseteq K \setminus KE_\phi$ if and only if (K_3).

Proof. It follows from the fact that $KE_\phi = \{\omega' \in \Omega : \kappa\phi \in \omega'\} = E_{\kappa\phi}$, and $\backslash KE_\phi = E_{\neg\kappa\phi}$. $\square$

Lemma 3. *Consider $\Omega \subseteq \Omega_2$, and let (K_E) hold in Ω. Then $\Omega \subseteq \Omega_4$.*

Proof. Consider $\kappa\phi \in \omega$. Then it follows from Proposition 2 that $\omega \in KE_\phi$, implying that $P(\omega) \subseteq E_\phi$. It follows from Samet [18] that $P(\omega') \subseteq P(\omega)$, for every $\omega' \in P(\omega)$, implying that $\omega' \in KE_\phi$, for every $\omega' \in P(\omega)$. Thus $\omega' \in \setminus K \setminus E_\phi$, for every $\omega' \in P(\omega)$. Hence $\omega \in K \setminus K \setminus E_\phi$. Finally it is follows from (K_E) that $K \setminus K \setminus E_\phi = E_{\kappa\neg\kappa\neg\phi}$, which concludes the proof. □

Proof of Proposition 3. We know that a possibility correspondence is partitional if and only if (a) $\omega \in P(\omega)$, for every $\omega \in \Omega$, and (b) $P(\omega) = P(\omega')$, for every $\omega' \in P(\omega)$, and every $\omega \in \Omega$ (Rubinstein [17]). It follows from Samet [18] that (a) holds if and only if (K_1). Supposing that $KE_\phi \subseteq KKE_\phi$ holds, we are going to show that (b) holds if and only if $\setminus KE_\phi \subseteq K \setminus KE_\phi$.
[⇒] It follows from $\omega \in KP(\omega) \subseteq KKP(\omega)$ that $P(\omega) \subseteq KP(\omega)$. Let $\omega' \in P(\omega)$. Then $\omega' \in KP(\omega)$, and therefore $P(\omega') \subseteq P(\omega)$. Suppose now that $P(\omega) \nsubseteq P(\omega')$, which is equivalent to $\omega \in \setminus KP(\omega') \subseteq K \setminus KP(\omega')$. Thus $P(\omega) \subseteq \setminus KP(\omega')$. Since $\omega' \in P(\omega)$, it follows that $P(\omega') \nsubseteq P(\omega')$, which is a contradiction. Therefore $P(\omega) \subseteq P(\omega')$, which completes the proof.
[⇐] Take some $E \subseteq \Omega$ such that $\omega \in \setminus KE$, which is equivalent to $P(\omega) \nsubseteq E$. Since $P(\omega') = P(\omega)$ for every $\omega' \in P(\omega)$, it follows that $P(\omega') \nsubseteq E$, implying that $P(\omega) \subseteq \{\omega' \in \Omega : P(\omega') \nsubseteq E\} = \setminus KE$. Hence, $\omega \in K \setminus KE$. □

Proof of Proposition 4. We know then that P is partitional if and only if $\omega \in P(\omega)$, for every $\omega \in \Omega$, $KE_\phi \subseteq KKE_\phi$, and $\setminus KE_\phi \subseteq K \setminus KE_\phi$, for every $\phi \in \Phi$. Then, the proof follows from $\Omega \subseteq \Omega_2$, and Lemmas 2 and 3. □