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Using the Matrix-Analytic Approach**

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# PRICING SYNTHETIC CDO TRANCHES IN A MODEL WITH DEFAULT CONTAGION USING THE MATRIX-ANALYTIC APPROACH

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ABSTRACT. We value synthetic CDO tranche spreads, index CDS spreads,  $k^{th}$ -to-default swap spreads and tranchelets in an intensity-based credit risk model with default contagion. The default dependence is modelled by letting individual intensities jump when other defaults occur. The model is reinterpreted as a Markov jump process. This allows us to use a matrix-analytic approach to derive computationally tractable closed-form expressions for the credit derivatives that we want to study. Special attention is given to homogenous portfolios. For a fixed maturity of five years, such a portfolio is calibrated against CDO tranche spreads, index CDS spread and the average CDS and FtD spreads, all taken from the iTraxx Europe series. After the calibration, which renders perfect fits, we compute spreads for tranchelets and  $k^{th}$ -to-default swap spreads for different subportfolios of the main portfolio. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios.

## 1. INTRODUCTION

In recent years the market for synthetic CDO tranches and index CDS-s, which are derivatives with a payoff linked to the credit loss in a portfolio of CDS-s, have seen a rapid growth and increased liquidity. This has been followed by an intense research for understanding and modelling the main feature driving these products, namely default dependence.

In this paper we derive computationally tractable closed-form expressions for synthetic CDO tranche spreads and index CDS spreads. This is done in an intensity based model where default dependencies among obligors are expressed in an intuitive, direct and compact way. The financial interpretation is that the individual default intensities are constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. This phenomena is often called default contagion. The above model is then reinterpreted

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in terms of a Markov jump process. This interpretation makes it possible to use a matrix-analytic approach to derive practical formulas for CDO tranche spreads and index CDS spreads. Our approach is the same as in [15] and [17] where the authors study aspects of  $k^{\text{th}}$ -to default spreads in nonsymmetric as well as in symmetric portfolios. The contribution of this paper is a continuation of this technique to synthetic CDO tranches and index CDS-s.

Except for [15] and [17], the methods presented in [2], [4], [7], [8], [9], [11], [12], Section 5.9 in [22] and Subsection 9.8.3 in [23], are currently closest to the approach of this article. The framework used here (and in [15] and [17]) is the same as in [11], [12] and is related to [2], [4]. The main differences are that [11], [12] use time-varying parameters in their practical examples and then solve the corresponding Chapman-Kolmogorov equation using numerical methods for ODE-systems. Furthermore, in [12], the authors also consider numerical examples where the portfolio is split into homogeneous groups with default contagion both within each group and between groups. [4] use Monte Carlo simulations to calibrate and price the instruments.

Default contagion in an intensity based setting have previously also been studied in for example [1], [3], [6], [13], [14], [19], [21], [25], [26] and [27]. The material in all these papers and books are related to the results discussed here.

This paper is organized as follows. In Section 2 we give an introduction to synthetic CDO tranches and index CDS-s which motivates results and introduces notation needed in the sequel. Section 3 presents the intensity-based model for default contagion. Using a result from [17], the model is reinterpreted in terms of a Markov jump process. The results in Section 4, convenient analytical formulas for synthetic CDO tranche spreads and index CDS spreads, are the main theoretical contribution in this paper. We assume that the recovery rates are deterministic and that the interest rate is constant. In Section 5 we apply the results from Section 4 to a homogenous model. Then, in Section 6, for a fixed maturity of five years, this portfolio is calibrated against CDO tranche spreads, the index CDS spread and the average CDS and FtD spreads, all taken from the iTraxx series, resulting in perfect fits. After the calibration, we compute  $k^{\text{th}}$ -to-default swap spreads for different subportfolios of the main portfolio. This problem is slightly different from the corresponding one in previous studies, e.g. [15] and [17], since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio. Further, we compute spreads on tranchelets which are nonstandard CDO tranches with smaller loss-intervals than standardized tranches. We also investigate implied tranche-losses and the implied loss distribution in the calibrated portfolios. The final section, Section 7 summarizes and discusses the results.

## 2. VALUATION OF SYNTHETIC CDO TRANCHE SPREADS AND INDEX CDS SPREADS

In this section we give a short description of tranche spreads in synthetic CDO-s and of index CDS spreads. It is independent of the underlying model for the default times and introduces notation needed later on. At the end of the section we give a technical

motivation for the main purpose of this article, which, roughly speaking, is to derive practical formulas for functions of the credit loss in a portfolio.

**2.1. The cash-flows in a synthetic CDO.** In this section and in the sequel all computations are assumed to be made under a risk-neutral martingale measure  $\mathbb{P}$ . Typically such a  $\mathbb{P}$  exists if we rule out arbitrage opportunities. Further, we assume that risk-free interest rate,  $r_t$  is deterministic.

A synthetic CDO is defined for a portfolio consisting of  $m$  single-name CDS's on obligors with default times  $\tau_1, \tau_2, \dots, \tau_m$  and recovery rates  $\phi_1, \phi_2, \dots, \phi_m$ . It is standard to assume that the nominal values are the same for all obligors, denoted by  $N$ . The accumulated credit loss  $L_t$  at time  $t$  for this portfolio is

$$L_t = \sum_{i=1}^m N(1 - \phi_i) 1_{\{\tau_i \leq t\}}. \quad (2.1.1)$$

We will without loss of generality express the loss  $L_t$  in percent of the nominal portfolio value at  $t = 0$ . For example, if all obligors in the portfolio have the same constant recovery rate  $\phi$ , then  $L_{T_k} = k(1 - \phi)/m$  where  $T_1 < \dots < T_k$  is the ordering of  $\tau_1, \tau_2, \dots, \tau_m$ .

A CDO is specified by the attachment points  $0 = k_0 < k_1 < k_2 < \dots < k_\kappa = 1$  with corresponding tranches  $[k_{\gamma-1}, k_\gamma]$ . The financial instrument that constitutes tranche  $\gamma$  with maturity  $T$  is a bilateral contract where the protection seller **B** agrees to pay the protection buyer **A**, all losses that occurs in the interval  $[k_{\gamma-1}, k_\gamma]$  derived from  $L_t$  up to time  $T$ . The payments are made at the corresponding default times, if they arrive before  $T$ , and at  $T$  the contract ends. The expected value of this payment is called the *protection leg*, denoted by  $V_\gamma(T)$ . As compensation for this, **A** pays **B** a periodic fee proportional to the current outstanding (possible reduced due to losses) value on tranche  $\gamma$  up to time  $T$ . The expected value of this payment scheme constitutes the *premium leg* denoted by  $W_\gamma(T)$ . The accumulated loss  $L_t^{(\gamma)}$  of tranche  $\gamma$  at time  $t$  is

$$L_t^{(\gamma)} = (L_t - k_{\gamma-1}) 1_{\{L_t \in [k_{\gamma-1}, k_\gamma]\}} + (k_\gamma - k_{\gamma-1}) 1_{\{L_t > k_\gamma\}}. \quad (2.1.2)$$

Let  $B_t = \exp\left(-\int_0^t r_s ds\right)$  denote the discount factor where  $r_t$  is the risk-free interest rate. The protection leg for tranche  $\gamma$  is then given by

$$V_\gamma(T) = \mathbb{E} \left[ \int_0^T B_t dL_t^{(\gamma)} \right] = B_T \mathbb{E} \left[ L_T^{(\gamma)} \right] + \int_0^T r_t B_t \mathbb{E} \left[ L_t^{(\gamma)} \right] dt,$$

where we have used integration by parts for Lebesgue-Stieltjes measures together with Fubini-Tonelli and the fact that  $r_t$  is deterministic. Further, if the premiums are paid at  $0 < t_1 < t_2 < \dots < t_{n_T} = T$  and if we ignore the accrued payments at defaults, then the premium leg is given by

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n} \left( \Delta k_\gamma - \mathbb{E} \left[ L_{t_n}^{(\gamma)} \right] \right) \Delta_n$$

where  $\Delta_n = t_n - t_{n-1}$  denote the times between payments (measured in fractions of a year) and  $\Delta k_\gamma = k_\gamma - k_{\gamma-1}$  is the nominal size of tranche  $\gamma$  (as a fraction of the total nominal value of the portfolio). The constant  $S_\gamma(T)$  is called the spread of tranche  $\gamma$  and is determined so that the value of the premium leg equals the value of the corresponding protection leg.

**2.2. The tranche spreads.** By definition, the constant  $S_\gamma(T)$  is determined at  $t = 0$  so that  $V_\gamma(T) = W_\gamma(T)$ , that is, so that the value of the premium leg agrees with the corresponding protection leg. Furthermore, for the first tranche, often denoted the *equity* tranche,  $S_1(T)$  is set to 500 bp and a so called *up-front* fee  $S_1^{(u)}(T)$  is added to the premium leg so that  $V_1(T) = S_1^{(u)}(T)k_1 + W_1(T)$ . Hence, we get that

$$S_\gamma(T) = \frac{B_T \mathbb{E} [L_T^{(\gamma)}] + \int_0^T r_t B_t \mathbb{E} [L_t^{(\gamma)}] dt}{\sum_{n=1}^{n_T} B_{t_n} (\Delta k_\gamma - \mathbb{E} [L_{t_n}^{(\gamma)}]) \Delta_n} \quad \gamma = 2, \dots, \kappa$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1} \left[ B_T \mathbb{E} [L_T^{(1)}] + \int_0^T r_t B_t \mathbb{E} [L_t^{(1)}] dt - 0.05 \sum_{n=1}^{n_T} B_{t_n} (\Delta k_1 - \mathbb{E} [L_{t_n}^{(1)}]) \Delta_n \right].$$

The spreads  $S_\gamma(T)$  are quoted in bp per annum while  $S_1^{(u)}(T)$  is quoted in percent per annum. Note that spreads are independent of the nominal size of the portfolio.

**2.3. The index CDS spread.** Consider the same synthetic CDO as above. An index CDS with maturity  $T$ , has almost the same structure as a corresponding CDO tranche, but with two main differences. First, the protection is on *all* credit losses that occurs in the CDO portfolio up to time  $T$ , so in the protection leg, the tranche loss  $L_t^{(\gamma)}$  is replaced by the total loss  $L_t$ . Secondly, in the premium leg, the spread is paid on a notional proportional to the number of obligors left in the portfolio at each payment date. Thus, if  $N_t$  denotes the number of obligors that have defaulted up to time  $t$ , i.e  $N_t = \sum_{i=1}^m \mathbf{1}_{\{\tau_i \leq t\}}$ , then the index CDS spread  $S(T)$  is paid on the notional  $(1 - \frac{N_t}{m})$ . Since the rest of the contract has the same structure as a CDO tranche, the value of the premium leg  $W(T)$  is

$$W(T) = S(T) \sum_{n=1}^{n_T} B_{t_n} \left( 1 - \frac{1}{m} \mathbb{E} [N_{t_n}] \right) \Delta_n$$

and the value of the protection leg,  $V(T)$ , is given by  $V(T) = B_T \mathbb{E} [L_T] + \int_0^T r_t B_t \mathbb{E} [L_t] dt$ . The index CDS spread  $S(T)$  is determined so that  $V(T) = W(T)$  which implies

$$S(T) = \frac{B_T \mathbb{E} [L_T] + \int_0^T r_t B_t \mathbb{E} [L_t] dt}{\sum_{n=1}^{n_T} B_{t_n} \left( 1 - \frac{1}{m} \mathbb{E} [N_{t_n}] \right) \Delta_n} \quad (2.3.1)$$

where  $\frac{1}{m} \mathbb{E} [N_t] = \frac{1}{1-\phi} \mathbb{E} [L_t]$  if  $\phi_1 = \phi_2 = \dots = \phi_m = \phi$ . The spread  $S(T)$  is quoted in bp per annum and is independent of the nominal size of the portfolio.

**2.4. The expected tranche losses.** From Subsection 2.2 we see that to compute tranche spreads we have to compute  $\mathbb{E} \left[ L_t^{(\gamma)} \right]$ , that is, the expected loss of the tranche  $[k_{\gamma-1}, k_\gamma]$  at time  $t$ . If we let  $F_{L_t}(x) = \mathbb{P} [L_t \leq x]$  then (2.1.2) implies that

$$\mathbb{E} \left[ L_t^{(\gamma)} \right] = (k_\gamma - k_{\gamma-1}) \mathbb{P} [L_t > k_\gamma] + \int_{k_{\gamma-1}}^{k_\gamma} (x - k_{\gamma-1}) dF_{L_t}(x). \quad (2.4.1)$$

Hence, in order to compute  $\mathbb{E} \left[ L_t^{(\gamma)} \right]$  and  $\mathbb{E} [L_t]$  and we must know the loss distribution  $F_{L_t}(x)$  at time  $t$ . Furthermore, if the recoveries are nonhomogeneous, then to determine the index CDS spread, we also must compute  $\mathbb{E} [N_{t_n}]$ , which is equivalent to finding the default distributions  $\mathbb{P} [\tau_i \leq t]$  for all obligors, or alternatively determining the distributions  $\mathbb{P} [T_k \leq t]$  for all ordered default times  $T_k$ .

To find analytical expressions for expected tranche losses, expected losses, and thus for tranche spreads and index CDS spread, is the main objective in this paper.

### 3. INTENSITY BASED MODELS REINTERPRETED AS MARKOV JUMP PROCESSES

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then translated into a Markov jump process. This makes it possible to use a matrix-analytic approach to derive computationally convenient formulas for CDO tranche spreads, index CDS spreads, single-name CDS spreads and  $k^{\text{th}}$ -to-default spreads. The model presented here is identical to the setup in [17] where the authors study aspects of  $k^{\text{th}}$ -to-default spreads in nonsymmetric as well as in symmetric portfolios. In this paper we focus on synthetic CDO tranches, index CDS and  $k^{\text{th}}$ -to-default swaps on subportfolios to the CDO portfolio.

With  $\tau_1, \tau_2, \dots, \tau_m$  default times as above, define the point process  $N_{t,i} = 1_{\{\tau_i \leq t\}}$  and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma (N_{s,i}; s \leq t), \quad \mathcal{F}_t = \bigvee_{i=1}^m \mathcal{F}_{t,i}.$$

Let  $\lambda_{t,i}$  be the  $\mathcal{F}_t$ -intensity of the point processes  $N_{t,i}$ . Below, we for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times  $\{\tau_i\}$  have intensities  $\{\lambda_{t,i}\}$ . The model studied in this paper is specified by requiring that the default intensities have the form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \quad \tau_i \geq t, \quad (3.1)$$

and  $\lambda_{t,i} = 0$  for  $t > \tau_i$ . Further,  $a_i \geq 0$  and  $b_{i,j}$  are constants such that  $\lambda_{t,i}$  is non-negative.

The financial interpretation of (3.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor  $i$  jumps by an amount  $b_{i,j}$  if it is obligor  $j$  which has defaulted. Thus a positive  $b_{i,j}$  means that obligor  $i$  is put at higher risk by the default of obligor  $j$ , while a negative  $b_{i,j}$  means that obligor  $i$  in fact

benefits from the default of  $j$ , and finally  $b_{i,j} = 0$  if obligor  $i$  is unaffected by the default of  $j$ .

Equation (3.1) determines the default times through their intensities. However, the expressions for the loss and tranche losses are in terms of their joint distributions. It is by no means obvious how to go from one to the other. Here we will use the following result, proved in [17].

**Proposition 3.1.** *There exists a Markov jump process  $(Y_t)_{t \geq 0}$  on a finite state space  $\mathbf{E}$  and a family of sets  $\{\Delta_i\}_{i=1}^m$  such that the stopping times*

$$\tau_i = \inf \{t > 0 : Y_t \in \Delta_i\}, \quad i = 1, 2, \dots, m,$$

*have intensities (3.1). Hence, any distribution derived from the multivariate stochastic vector  $(\tau_1, \tau_2, \dots, \tau_m)$  can be obtained from  $\{Y_t\}_{t \geq 0}$ .*

Each state  $\mathbf{j}$  in  $\mathbf{E}$  is of the form  $\mathbf{j} = \{j_1, \dots, j_k\}$  which is a subsequence of  $\{1, \dots, m\}$  consisting of  $k$  integers, where  $1 \leq k \leq m$ . The interpretation is that on  $\{j_1, \dots, j_k\}$  the obligors in the set have defaulted. The Markov jump process  $Y_t$  on  $\mathbf{E}$  is specified by making  $\{1, \dots, m\}$  absorbing and starting in  $\{0\}$ .

In this paper, Proposition 3.1 is throughout used for computing distributions. However, we still use Equation (3.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive. In the sequel, we let  $\mathbf{Q}$  and  $\boldsymbol{\alpha}$  denote the generator and initial distribution on  $\mathbf{E}$  for the Markov jump process in Proposition 3.1. The generator  $\mathbf{Q}$  is found by using the structure of  $\mathbf{E}$ , the definition of the states  $\mathbf{j}$ , and Equation (3.1), see [17]. By construction  $\boldsymbol{\alpha} = (1, 0, \dots, 0)$ . Further, if  $\mathbf{j}$  belongs to  $\mathbf{E}$  then  $\mathbf{e}_{\mathbf{j}}$  denotes a column vector in  $\mathbb{R}^{|\mathbf{E}|}$  where the entry at position  $\mathbf{j}$  is 1 and the other entries are zero. From Markov theory we know that  $\mathbb{P}[Y_t = \mathbf{j}] = \boldsymbol{\alpha} e^{\mathbf{Q}t} \mathbf{e}_{\mathbf{j}}$  where  $e^{\mathbf{Q}t}$  is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of  $\mathbf{Q}$ .

#### 4. USING THE MATRIX-ANALYTIC APPROACH TO FIND CDO TRANCHE SPREADS AND INDEX CDS SPREADS

In this section we derive practical formulas for CDO tranche spreads and index CDS spreads. This is done under (3.1) together with the standard assumption of deterministic recovery rates and constant interest rate  $r$ . Although the derivation is done in an inhomogeneous portfolio, we will in Section 5 show that these formulas are almost the same in a homogeneous model.

The following observation is a key to all results in this article. If the obligors in a portfolio satisfy (3.1) and have deterministic recoveries, then Proposition 3.1 implies that the corresponding loss  $L_t$  can be represented as a functional of the Markov jump process  $Y_t$ ,  $L_t = L(Y_t)$  where the mapping  $L$  goes from  $\mathbf{E}$  to all possible loss-outcomes determined via (2.1.1). For example, if  $\mathbf{j} \in \mathbf{E}$  where  $\mathbf{j} = \{j_1, \dots, j_k\}$  then  $L(\mathbf{j}) = \frac{1}{m} \sum_{n=1}^k (1 - \phi_{j_n})$ . The range of  $L$  is a finite set since the recoveries are deterministic. This implies that for any mapping  $g(x)$  on  $\mathbb{R}$  and a set  $A$  in  $[0, \infty)$ , we have

$$\int_A g(x) dF_{L_t}(x) = \boldsymbol{\alpha} e^{\mathbf{Q}t} \mathbf{h}(g, A)$$

where  $\mathbf{h}(g, A)$  is a column vector in  $\mathbb{R}^{|\mathbf{E}|}$  defined by  $\mathbf{h}(g, A)_j = g(L(\mathbf{j}))1_{\{L(\mathbf{j}) \in A\}}$ . From this we obtain the following easy lemma, which is stated since it provides notation which is needed later on.

**Lemma 4.1.** *Consider a synthetic CDO on a portfolio with  $m$  obligors that satisfy (3.1). Then, with notation as above,*

$$\mathbb{E} \left[ L_t^{(\gamma)} \right] = \boldsymbol{\alpha} e^{\mathbf{Q}t} \boldsymbol{\ell}^{(\gamma)}, \quad \mathbb{E} [L_t] = \boldsymbol{\alpha} e^{\mathbf{Q}t} \boldsymbol{\ell} \quad \text{and} \quad \mathbb{E} [N_t] = \boldsymbol{\alpha} e^{\mathbf{Q}t} \sum_{i=1}^m \mathbf{h}^{(i)}$$

where  $\boldsymbol{\ell}^{(\gamma)}$  is a column vector in  $\mathbb{R}^{|\mathbf{E}|}$  defined by

$$\boldsymbol{\ell}_j^{(\gamma)} = \begin{cases} 0 & \text{if } L(\mathbf{j}) < k_{\gamma-1} \\ L(\mathbf{j}) - k_{\gamma-1} & \text{if } L(\mathbf{j}) \in [k_{\gamma-1}, k_\gamma] \\ \Delta k_\gamma & \text{if } L(\mathbf{j}) > k_\gamma \end{cases} \quad (4.1)$$

and  $L$  is the mapping such that  $L_t = L(Y_t)$ . Furthermore,  $\boldsymbol{\ell}$  and  $\mathbf{h}^{(i)}$  are column vectors in  $\mathbb{R}^{|\mathbf{E}|}$  defined by  $\boldsymbol{\ell}_j = L(\mathbf{j})$  and  $\mathbf{h}_j^{(i)} = 1_{\{j \in \Delta_i\}}$  where the sets  $\Delta_i$  are as in Proposition 3.1.

We now present the main results of this paper.

**Proposition 4.2.** *Consider a synthetic CDO on a portfolio with  $m$  obligors that satisfy (3.1) and assume that the interest rate  $r$  is constant. Then, with notation as above,*

$$S_\gamma(T) = \frac{(\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r) \boldsymbol{\ell}^{(\gamma)}}{\sum_{n=1}^{n_T} e^{-rt_n} (\Delta k_\gamma - \boldsymbol{\alpha} e^{\mathbf{Q}t_n} \boldsymbol{\ell}^{(\gamma)}) \Delta_n} \quad \gamma = 2, \dots, \kappa \quad (4.2)$$

and

$$S_1^{(u)}(T) = \frac{1}{k_1} \left( \boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r + 0.05 \sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\mathbf{Q}t_n} e^{-rt_n} \Delta_n \right) \boldsymbol{\ell}^{(1)} - 0.05 \sum_{n=1}^{n_T} e^{-rt_n} \Delta_n \quad (4.3)$$

where

$$\mathbf{R}(0, T) = \int_0^T e^{(\mathbf{Q}-r\mathbf{I})t} dt = (e^{\mathbf{Q}T} e^{-rT} - \mathbf{I}) (\mathbf{Q} - r\mathbf{I})^{-1}. \quad (4.4)$$

Furthermore,

$$S(T) = \frac{(\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r) \boldsymbol{\ell}}{\sum_{n=1}^{n_T} e^{-rt_n} (1 - \boldsymbol{\alpha} e^{\mathbf{Q}t_n} \widehat{\boldsymbol{\ell}}) \Delta_n} \quad (4.5)$$

where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi} \boldsymbol{\ell} & \text{if } \phi_1 = \phi_2 = \dots = \phi_m = \phi \\ \frac{1}{m} \sum_{i=1}^m \mathbf{h}^{(i)} & \text{otherwise} \end{cases}. \quad (4.6)$$

*Proof.* Since  $r_t = r$ , using Lemma 4.1 we have that

$$\int_0^T r_t B_t \mathbb{E} \left[ L_t^{(\gamma)} \right] dt = \boldsymbol{\alpha} \int_0^T e^{(\mathbf{Q}-r\mathbf{I})t} dt \boldsymbol{\ell}^{(\gamma)} r = \boldsymbol{\alpha} \mathbf{R}(0, T) \boldsymbol{\ell}^{(\gamma)} r$$



where  $\mathbf{R}(0, T)$  is given by (4.4). So by Lemma 4.1 again, we get

$$V_\gamma(T) = B_T \mathbb{E} \left[ L_T^{(\gamma)} \right] + \int_0^T r_t B_t \mathbb{E} \left[ L_t^{(\gamma)} \right] dt = (\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r) \boldsymbol{\ell}^{(\gamma)}$$

and

$$W_\gamma(T) = S_\gamma(T) \sum_{n=1}^{n_T} B_{t_n} \left( \Delta k_\gamma - \mathbb{E} \left[ L_{t_n}^{(\gamma)} \right] \right) \Delta_n = S_\gamma(T) \sum_{n=1}^{n_T} e^{-rt_n} \left( \Delta k_\gamma - \boldsymbol{\alpha} e^{\mathbf{Q}t_n} \boldsymbol{\ell}^{(\gamma)} \right) \Delta_n.$$

Recall that for all tranches  $\gamma$ , except for the equity tranche, the spreads  $S_\gamma(T)$  are determined so that  $V_\gamma(T) = W_\gamma(T)$ . Thus, the equations above prove (4.2). Furthermore, for the equity tranche,  $S_1(T)$  is set to 500 bp and the up-front premium  $S_1^{(u)}(T)$  is determined so that  $V_1(T) = S_1^{(u)}(T)k_1 + W_1(T)$ . The expressions for  $V_1(T)$  and  $W_1(T)$  together with the fact that  $\Delta k_1 = k_1$  then imply that  $S_1^{(u)}(T)$  is given by

$$\begin{aligned} S_1^{(u)}(T) &= \frac{1}{k_1} \left[ B_T \mathbb{E} \left[ L_T^{(1)} \right] + \int_0^T r_t B_t \mathbb{E} \left[ L_t^{(1)} \right] dt - 0.05 \sum_{n=1}^{n_T} B_{t_n} \left( \Delta k_1 - \mathbb{E} \left[ L_{t_n}^{(1)} \right] \right) \Delta_n \right] \\ &= \frac{1}{k_1} \left[ (\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r) \boldsymbol{\ell}^{(1)} - 0.05 \sum_{n=1}^{n_T} e^{-rt_n} \left( \Delta k_1 - \boldsymbol{\alpha} e^{\mathbf{Q}t_n} \boldsymbol{\ell}^{(1)} \right) \Delta_n \right] \\ &= \frac{1}{k_1} \left( \boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r + 0.05 \sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\mathbf{Q}t_n} e^{-rt_n} \Delta_n \right) \boldsymbol{\ell}^{(1)} - 0.05 \sum_{n=1}^{n_T} e^{-rt_n} \Delta_n \end{aligned}$$

which establish (4.3). Finally, to find expressions for the index CDS spreads  $S(T)$ , recall that this contract is almost identical to a CDO tranche (see (2.3.1)), with the differences that  $\boldsymbol{\ell}^{(\gamma)}$  is replaced by  $\boldsymbol{\ell}$  in the protection leg, and in the premium leg  $\Delta k_\gamma$  is replaced by 1 and  $\boldsymbol{\ell}^{(\gamma)}$  by  $\widehat{\boldsymbol{\ell}}$ , where

$$\widehat{\boldsymbol{\ell}} = \begin{cases} \frac{1}{1-\phi} \boldsymbol{\ell} & \text{if } \phi_1 = \phi_2 = \dots = \phi_m = \phi \\ \frac{1}{m} \sum_{i=1}^m \mathbf{h}^{(i)} & \text{otherwise} \end{cases}$$

which proves (4.5) and (4.6).  $\square$

The message of Proposition 4.2 is that under (3.1), computations of CDO tranche spreads and index CDS spreads are reduced to compute the matrix exponential. Finding the generator  $\mathbf{Q}$  and column vectors  $\boldsymbol{\ell}^{(\gamma)}$ ,  $\boldsymbol{\ell}$ ,  $\widehat{\boldsymbol{\ell}}$  are straightforward and the matrix  $(\mathbf{Q} - r\mathbf{I})$  is invertible since it is upper diagonal with strictly negative diagonal elements, see [17]. Computing  $e^{\mathbf{Q}t}$  efficiently is a numerical issue, which for large state spaces requires special treatment, see [17]. For small state spaces, typically less than 150 states, the task is straightforward using standard mathematical software. Several computational shortcuts are possible in Proposition 4.2. The quantities  $\boldsymbol{\ell}^{(\gamma)}$ ,  $\boldsymbol{\ell}$  and  $\widehat{\boldsymbol{\ell}}$  do not depend on the parametrization, and hence only have to be computed once. The row vectors  $\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha} \mathbf{R}(0, T) r$  and  $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\mathbf{Q}t_n} e^{-rt_n} \Delta_n$  are the same for all CDO tranche spreads and index CDS spreads and hence only have to be computed once for each parametrization determined by (3.1). In

particular note that  $\sum_{n=1}^{n_T} \alpha e^{Q t_n} e^{-r t_n} \Delta_n$  and  $(Q - rI)^{-1}$  also appears in the expressions for single-name CDS spreads and  $k^{\text{th}}$ -to-default spreads studied in [17].

In a nonhomogeneous portfolio we have  $|\mathbf{E}| = 2^m$  which in practice will force us to work with portfolios of size  $m$  less or equal to 25, say ([17] used  $m = 15$ ). Standard synthetic CDO portfolios typically contains 125 obligors so we will therefore, in Section 5 below, consider a special case of (3.1) which leads to a symmetric portfolio where the state space  $\mathbf{E}$  can be simplified to make  $|\mathbf{E}| = m + 1$ . This allows us to practically work with the Markov setup in Proposition 4.2 for large  $m$ , where  $m \geq 125$  with no further complications. Using homogeneous credit portfolio models when pricing CDO tranches is currently standard in almost all credit literature today.

## 5. A HOMOGENEOUS PORTFOLIO

In this section we apply the results from Section 4 to a homogenous portfolio. First, Subsection 5.1 introduces a symmetric model and shows how it can be applied to price CDO tranche spreads and index CDS spreads. Subsection 5.2 presents formulas for the single-name CDS spread in this model. Finally, Subsection 5.3 is devoted to formulas for  $k^{\text{th}}$ -to-default swaps on subportfolios of the main portfolio. This problem is slightly different from the corresponding task in previous studies, e.g. [15] and [17], since the obligors undergo default contagion both from the subportfolio and from obligors outside the subportfolio, in the main portfolio.

**5.1. The homogeneous model for CDO tranches and index CDS-s.** In this subsection we use the results from Section 4 to compute CDO tranche spreads and index CDS spreads in a totally symmetric model. We consider a special case of (3.1) where all obligors have the same default intensities  $\lambda_{t,i} = \lambda_t$  specified by parameters  $a$  and  $b_1, \dots, b_m$ , as

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \quad (5.1.1)$$

where  $\{T_k\}$  is the ordering of the default times  $\{\tau_i\}$  and  $\phi_1 = \dots = \phi_m = \phi$  where  $\phi$  is constant. In this model the obligors are exchangeable. The parameter  $a$  is the base intensity for each obligor  $i$ , and given that  $\tau_i > T_k$ , then  $b_k$  is how much the default intensity for each remaining obligor jump at default number  $k$  in the portfolio. We start with the simpler version of Proposition 3.1.

**Corollary 5.1.** *There exists a Markov jump process  $(Y_t)_{t \geq 0}$  on a finite state space  $\mathbf{E} = \{0, 1, 2, \dots, m\}$ , such that the stopping times*

$$T_k = \inf \{t > 0 : Y_t = k\}, \quad k = 1, \dots, m$$

*are the ordering of  $m$  exchangeable stopping times  $\tau_1, \dots, \tau_m$  with intensities (5.1.1).*

*Proof.* If  $\{T_k\}$  is the ordering of  $m$  default times  $\{\tau_i\}$  with default intensities  $\{\lambda_{t,i}\}$ , then the arrival intensity  $\lambda_t^{(k)}$  for  $T_k$  is zero outside of  $\{T_{k-1} \leq t < T_k\}$ , otherwise

$$\lambda_t^{(k)} = \left( \sum_{i=1}^m \lambda_{t,i} \right) 1_{\{T_{k-1} \leq t < T_k\}}. \quad (5.1.2)$$

Hence, since  $\lambda_{t,i} = \lambda_t$  for every obligor  $i$  where  $\tau_i \geq t$ , (5.1.2) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m - k + 1}, \quad k = 1, \dots, m. \quad (5.1.3)$$

Now, let  $(Y_t)_{t \geq 0}$  be a Markov jump process on a finite state space  $\mathbf{E} = \{0, 1, 2, \dots, m\}$ , with generator  $\mathbf{Q}$  given by

$$\begin{aligned} \mathbf{Q}_{k,k+1} &= (m - k) \left( a + \sum_{j=1}^k b_j \right) \quad k = 0, 1, \dots, m - 1 \\ \mathbf{Q}_{k,k} &= -\mathbf{Q}_{k,k+1}, \quad k < m \quad \text{and} \quad \mathbf{Q}_{m,m} = 0 \end{aligned}$$

where the other entries in  $\mathbf{Q}$  are zero. The Markov process always starts in  $\{0\}$  so the initial distribution is  $\boldsymbol{\alpha} = (1, 0, \dots, 0)$ . Define the ordered stopping times  $\{T_k\}$  as

$$T_k = \inf \{t > 0 : Y_t = k\}, \quad k = 1, \dots, m.$$

Then, the intensity  $\lambda_t^{(k)}$  for  $T_k$  on  $\{T_{k-1} \leq t < T_k\}$  is given by  $\lambda_t^{(k)} = \mathbf{Q}_{k-1,k}$ . Further, we can without loss of generality assume that  $\{T_k\}$  is the ordering of  $m$  exchangeable default times  $\{\tau_i\}$ , with default intensities  $\lambda_{t,i} = \lambda_t$  for every obligor  $i$ . Hence, if  $\tau_i \geq t$ , (5.1.3) implies

$$\lambda_t 1_{\{T_{k-1} \leq t < T_k\}} = \frac{\lambda_t^{(k)}}{m - k + 1} = \frac{\mathbf{Q}_{k-1,k}}{m - k + 1} = a + \sum_{j=1}^{k-1} b_j, \quad k = 1, \dots, m$$

and since  $\lambda_t = \sum_{k=1}^m \lambda_t 1_{\{T_{k-1} \leq t < T_k\}}$ , it must hold that  $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$ , when  $\tau_i \geq t$ , which proves the corollary.  $\square$

By Corollary 5.1, the states in  $\mathbf{E}$  can be interpreted as the number of defaulted obligors in the portfolio.

Recall that the formulas for CDO tranche spreads and index CDS spreads in Proposition 4.2 were derived for an inhomogeneous portfolio with default intensities (3.1). However, it is easy to see that these formulas (with identical recoveries) also can be applied in a homogeneous model specified by (5.1.1), but with  $\boldsymbol{\ell}^{(\gamma)}$  and  $\boldsymbol{\ell}$  slightly refined to match the homogeneous state space  $\mathbf{E}$ . This refinement is shown in the following lemma.

**Lemma 5.2.** *Consider a portfolio with  $m$  obligors that all satisfy (5.1.1) and let  $\mathbf{E}$ ,  $\mathbf{Q}$  and  $\boldsymbol{\alpha}$  be as in Corollary 5.1. Then, (4.2), (4.3) and (4.5) hold, for*

$$\boldsymbol{\ell}_k^{(\gamma)} = \begin{cases} 0 & \text{if } k < n_l(k_{\gamma-1}) \\ k(1 - \phi)/m - k_{\gamma-1} & \text{if } n_l(k_{\gamma-1}) \leq k \leq n_u(k_{\gamma}) \\ \Delta k_{\gamma} & \text{if } k > n_u(k_{\gamma}) \end{cases} \quad (5.1.4)$$

where  $n_l(x) = \lceil xm/(1 - \phi) \rceil$  and  $n_u(x) = \lfloor xm/(1 - \phi) \rfloor$ . Furthermore,  $\ell_k = k(1 - \phi)/m$ .

*Proof.* Since  $L_t = L(Y_t)$  and due to the homogeneous structure, we have

$$\{L_t = k(1 - \phi)/m\} = \{Y_t = k\}$$

for each  $k$  in  $\mathbf{E}$ . Hence, the loss process  $L_t$  is in one-to-one correspondence with the process  $Y_t$ . Define  $n_l(x) = \lceil xm/(1 - \phi) \rceil$  and  $n_u(x) = \lfloor xm/(1 - \phi) \rfloor$ . That is,  $n_l(x)$  ( $n_u(x)$ ) is the smallest (biggest) integer bigger (smaller) or equal to  $xm/(1 - \phi)$ . These observations together with the expression for  $\ell^{(\gamma)}$  and  $\ell$  in Proposition 4.1, yield (5.1.4).  $\square$

In the homogeneous model given by (5.1.1), we have now determined all quantities needed to compute CDO tranche spreads and index CDS spreads as specified in Proposition 4.2.

**5.2. Pricing single-name CDS in a homogeneous model.** If  $F(t)$  is the distribution for  $\tau_i$ , which by exchangeability is the same for all obligors under (5.1.1), then the single-name CDS spread  $R(T)$  is given by (see e.g. [17])

$$R(T) = \frac{(1 - \phi) \int_0^T B_t dF(t)}{\sum_{n=1}^{n_T} \left( B_{t_n} \Delta_n (1 - F(t_n)) + \int_{t_{n-1}}^{t_n} B_t (t - t_{n-1}) dF(t) \right)} \quad (5.2.1)$$

where the rest of the notation are the same as in Section 2. Hence, to calibrate, or price single-name CDS-s under (5.1.1), we need the distribution  $\mathbb{P}[\tau_i > t]$  (identical for all obligors). This leads to the following lemma.

**Lemma 5.3.** *Consider  $m$  obligors that satisfy (5.1.1). Then, with notation as above*

$$\mathbb{P}[\tau_i > t] = \alpha e^{\mathcal{Q}t} \mathbf{g} \quad \text{and} \quad \mathbb{P}[T_k > t] = \alpha e^{\mathcal{Q}t} \mathbf{m}^{(k)}, \quad k = 1, \dots, m$$

where  $\mathbf{m}^{(k)}$  and  $\mathbf{g}$  are column vectors in  $\mathbb{R}^{|\mathbf{E}|}$  such that  $\mathbf{m}_j^{(k)} = 1_{\{j < k\}}$  and  $\mathbf{g}_j = 1 - j/m$ .

*Proof.* By the construction of  $T_k$  in Corollary 5.1, we have

$$\mathbb{P}[T_k > t] = \mathbb{P}[Y_t < k] = \sum_{j=0}^{k-1} \alpha e^{\mathcal{Q}t} \mathbf{e}_j = \alpha e^{\mathcal{Q}t} \mathbf{m}^{(k)} \quad \text{where} \quad \mathbf{m}_j^{(k)} = 1_{\{j < k\}}$$

for  $k = 1, \dots, m$ . Furthermore, due to the exchangeability,

$$\mathbb{P}[T_k > t] = \sum_{i=1}^m \mathbb{P}[T_k > t, T_k = \tau_i] = m \mathbb{P}[T_k > t, T_k = \tau_i]$$

so

$$\mathbb{P}[\tau_i > t] = \sum_{k=1}^m \mathbb{P}[T_k > t, T_k = \tau_i] = \sum_{k=1}^m \frac{1}{m} \mathbb{P}[T_k > t] = \alpha e^{\mathcal{Q}t} \sum_{k=1}^m \frac{1}{m} \mathbf{m}^{(k)} = \alpha e^{\mathcal{Q}t} \mathbf{g},$$

where  $\mathbf{g} = \frac{1}{m} \sum_{k=1}^m \mathbf{m}^{(k)}$ . Since  $\mathbf{m}_j^{(k)} = 1_{\{j < k\}}$  this implies that  $\mathbf{g}_j = 1 - j/m$  which concludes the proof of the lemma.  $\square$

A closed-form expression for  $R(T)$  is obtained by using Lemma 5.3 in (5.2.1). For ease of reference we exhibit the resulting formulas (proofs can be found in [15] or [16]).

**Proposition 5.4.** *Consider  $m$  obligors that all satisfies (5.1.1) and assume that the interest rate  $r$  is constant. Then, with notation as above*

$$R(T) = \frac{(1 - \phi)\boldsymbol{\alpha}(\mathbf{A}(0) - \mathbf{A}(T))\mathbf{g}}{\boldsymbol{\alpha}(\sum_{n=1}^{n_T} (\Delta_n e^{\mathbf{Q}t_n} e^{-rt_n} + \mathbf{C}(t_{n-1}, t_n)))\mathbf{g}}$$

where

$$\mathbf{C}(s, t) = s(\mathbf{A}(t) - \mathbf{A}(s)) - \mathbf{B}(t) + \mathbf{B}(s), \quad \mathbf{A}(t) = e^{\mathbf{Q}t}(\mathbf{Q} - r\mathbf{I})^{-1}\mathbf{Q}e^{-rt}$$

and

$$\mathbf{B}(t) = e^{\mathbf{Q}t}(t\mathbf{I} + (\mathbf{Q} - r\mathbf{I})^{-1})(\mathbf{Q} - r\mathbf{I})^{-1}\mathbf{Q}e^{-rt}.$$

For more on the CDS contract, see e.g [10], [15] or [23].

**5.3. Pricing  $k^{\text{th}}$ -to-default swaps on subportfolios in a homogeneous model.** Consider a homogenous portfolio defined by (5.1.1). Our goal in this subsection is to find expressions for  $k^{\text{th}}$ -to-default swap spreads on a subportfolio in the main portfolio. The difference in this approach, compared with for example [17] and [12] is that the obligors undergoes default contagion both from entities in the selected basket and from obligors outside the basket, but in the main portfolio.

Let  $\mathbf{s}$  be a subportfolio of the main portfolio, that is  $\mathbf{s} \subseteq \{1, 2, \dots, m\}$  and let  $|\mathbf{s}|$  denote the number of obligors in  $\mathbf{s}$  so  $|\mathbf{s}| \leq m$ . The market standard is  $|\mathbf{s}| = 5$ . If the recoveries are homogeneous, it is enough to find the distribution for the ordering of the default times in the basket. Hence, we seek the distributions of the ordered default times in  $\mathbf{s}$  denoted by  $\{T_k^{(\mathbf{s})}\}$ . The  $k^{\text{th}}$ -to-default swap spreads  $R_k^{(\mathbf{s})}(T)$  on  $\mathbf{s}$  are then given by (see e.g. [17])

$$R_k^{(\mathbf{s})}(T) = \frac{(1 - \phi) \int_0^T B_t dF_k^{(\mathbf{s})}(t)}{\sum_{n=1}^{n_T} \left( B_{t_n} \Delta_n (1 - F_k^{(\mathbf{s})}(t_n)) + \int_{t_{n-1}}^{t_n} B_t (t - t_{n-1}) dF_k^{(\mathbf{s})}(t) \right)} \quad (5.3.1)$$

where  $F_k^{(\mathbf{s})}(t) = \mathbb{P}[T_k^{(\mathbf{s})} \leq t]$  are the distribution functions for  $\{T_k^{(\mathbf{s})}\}$ . The rest of the notation are the same as in Section 2. In Theorem 5.5 below, we derive formulas for the survival distributions of  $\{T_k^{(\mathbf{s})}\}$ . This is done by using the exchangeability, the matrix-analytic approach and the fact that default times in  $\mathbf{s}$  always coincide with a subsequence of the default times in the main portfolio.

**Theorem 5.5.** *Consider a portfolio with  $m$  obligors that satisfy (5.1.1) and let  $\mathbf{s}$  be an arbitrary subportfolio with  $|\mathbf{s}|$  obligors. Then, with notation as above*

$$\mathbb{P}[T_k^{(\mathbf{s})} > t] = \boldsymbol{\alpha} e^{\mathbf{Q}t} \mathbf{m}^{k, \mathbf{s}} \quad \text{for } k = 1, 2, \dots, |\mathbf{s}| \quad (5.3.2)$$

where

$$\mathbf{m}_j^{k, \mathbf{s}} = \begin{cases} 1 & \text{if } j < k \\ 1 - \sum_{\ell=k}^{j \wedge |\mathbf{s}|} \frac{\binom{|\mathbf{s}|}{\ell} \binom{m-|\mathbf{s}|}{j-\ell}}{\binom{m}{j}} & \text{if } j \geq k. \end{cases} \quad (5.3.3)$$

*Proof.* The events  $\{T_\ell > t\}$  and  $\{T_k^{(\mathbf{s})} = T_\ell\}$  are independent where  $k \leq \ell \leq m - |\mathbf{s}| + k$ . To motivate this, note that since all obligors are exchangeable, the information  $\{T_k^{(\mathbf{s})} = T_\ell\}$  will not influence the event  $\{T_\ell > t\}$ . Thus,  $\mathbb{P}[T_\ell > t, T_k^{(\mathbf{s})} = T_\ell] = \mathbb{P}[T_\ell > t] \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell]$ . This observations together with Lemma 5.3 implies that

$$\begin{aligned} \mathbb{P}[T_k^{(\mathbf{s})} > t] &= \sum_{\ell=k}^{m-|\mathbf{s}|+k} \mathbb{P}[T_k^{(\mathbf{s})} > t, T_k^{(\mathbf{s})} = T_\ell] \\ &= \sum_{\ell=k}^{m-|\mathbf{s}|+k} \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell] \mathbb{P}[T_\ell > t] \\ &= \sum_{\ell=k}^{m-|\mathbf{s}|+k} \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell] \alpha e^{\mathbf{Q}t} \mathbf{m}^{(\ell)} = \alpha e^{\mathbf{Q}t} \mathbf{m}^{k,\mathbf{s}} \end{aligned}$$

where

$$\mathbf{m}^{k,\mathbf{s}} = \sum_{\ell=k}^{m-|\mathbf{s}|+k} \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell] \mathbf{m}^{(\ell)}.$$

Using this and the definition of  $\mathbf{m}_j^{(\ell)}$  renders

$$\mathbf{m}_j^{k,\mathbf{s}} = \begin{cases} 1 & \text{if } j < k \\ 1 - \sum_{\ell=k}^j \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell] & \text{if } j \geq k \end{cases}$$

and in order to compute  $\mathbf{m}_j^{k,\mathbf{s}}$  for  $j \geq k$ , note that

$$\bigcup_{\ell=k}^j \{T_k^{(\mathbf{s})} = T_\ell\} = \{k \leq N_j^{(\mathbf{s})} \leq j \wedge |\mathbf{s}|\}$$

where  $N_j^{(\mathbf{s})}$  is defined as  $N_j^{(\mathbf{s})} = \sup\{n : T_n^{(\mathbf{s})} \leq T_j\}$ , that is, the number of obligors that have defaulted in the subportfolio  $\mathbf{s}$  up to the  $j$ -th default in the main portfolio. Due to the exchangeability,  $N_j^{(\mathbf{s})}$  is a hypergeometric random variable with parameters  $m$ ,  $j$  and  $|\mathbf{s}|$ . Hence,

$$\sum_{\ell=k}^j \mathbb{P}[T_k^{(\mathbf{s})} = T_\ell] = \sum_{\ell=k}^{j \wedge |\mathbf{s}|} \mathbb{P}[N_j^{(\mathbf{s})} = \ell] = \sum_{\ell=k}^{j \wedge |\mathbf{s}|} \frac{\binom{|\mathbf{s}|}{\ell} \binom{m-|\mathbf{s}|}{j-\ell}}{\binom{m}{j}}.$$

which proves the theorem.  $\square$

Returning to  $k^{\text{th}}$ -to-default swap spreads, expressions for  $R_k^{(\mathbf{s})}(T)$  may be obtained by inserting (5.3.2) into (5.3.1). The notation and proof are the same as in Proposition 5.4

**Corollary 5.6.** *Consider a portfolio with  $m$  obligors that satisfy (5.1.1) and let  $\mathbf{s}$  be an arbitrary subportfolio with  $|\mathbf{s}|$  obligors. Assume that the interest rate  $r$  is constant. Then, with notation as above,*

$$R_k^{(\mathbf{s})}(T) = \frac{(1 - \phi)\boldsymbol{\alpha}(\mathbf{A}(0) - \mathbf{A}(T))\mathbf{m}^{k,\mathbf{s}}}{\boldsymbol{\alpha}(\sum_{n=1}^{n_T} (\Delta_n e^{\mathbf{Q}t_n} e^{-rt_n} + \mathbf{C}(t_{n-1}, t_n))\mathbf{m}^{k,\mathbf{s}})}, \quad k = 1, 2, \dots, |\mathbf{s}|.$$

For a more detailed description of  $k^{\text{th}}$ -to-default swap, see e.g. [10], [15], [17] or [23].

## 6. NUMERICAL STUDY OF A HOMOGENEOUS PORTFOLIO

In this section we calibrate the homogeneous portfolio to real market data on CDO tranches, index CDS-s, average single-name CDS spreads and average FtD-spreads (i.e. average  $1^{\text{th}}$ -to-default swaps). We match the theoretical spreads against the corresponding market spreads for individual default intensities given by (5.1.1). First, in Subsection 6.1 we give an outline of the calibration technique used in this paper. Then, in Subsection 6.2 we calibrate our model against an example studied in several articles, e.g [12] and [18], with data from iTraxx Europe, August 4, 2004. The iTraxx Europe spreads has changed drastically in the period between August 2004 and November 2006. We therefore recalibrate our model to a more recent data set, collected at November 28<sup>th</sup>, 2006. This second calibration also lends some confidence to the robustness of our model.

Having calibrated the portfolio, we can compute spreads for exotic credit derivatives, not liquidly quoted on the market, as well as other quantities relevant for credit portfolio management. In Subsection 6.3 we compute spreads for tranchelets, which are CDO tranches with smaller loss-intervals than standardized tranches. Subsection 6.4 investigates  $k^{\text{th}}$ -to-default swap spreads as function of the size of the underlying subportfolio in main calibrated portfolio. Continuing, Subsection 6.5 studies the implied expected loss in the portfolio and the implied expected tranche-losses. Finally, Subsection 6.6 is devoted to explore the implied loss-distribution as function of time.

**6.1. Some remarks on the calibration.** The symmetric model (5.1.1) can contain at most  $m$  different parameters. Our goal is to achieve a "perfect fit" with as many parameters as there are market spreads used in the calibration for a fixed maturity  $T$ . For a standard synthetic CDO such as the iTraxx Europe series, we can have 5 tranche spreads, the index CDS spread, the average single-name CDS spread and the average FtD spread. Hence, for calibration, there is at most 8 market prices with maturity  $T$  available. However, all of them do not have to be used. We make the following assumption on the parameters  $b_k$  for  $1 \leq k \leq m - 1$

$$b_k = \begin{cases} b^{(1)} & \text{if } 1 \leq k < \mu_1 \\ b^{(2)} & \text{if } \mu_1 \leq k < \mu_2 \\ \vdots & \\ b^{(c)} & \text{if } \mu_{c-1} \leq k < \mu_c = m \end{cases} \quad (6.1.1)$$

where  $1, \mu_1, \mu_2, \dots, \mu_c$  is an partition of  $\{1, 2, \dots, m\}$ . This means that all jumps in the intensity at the defaults  $1, 2, \dots, \mu_1 - 1$  are same and given by  $b^{(1)}$ , all jumps in the intensity

at the defaults  $\mu_1, \dots, \mu_2 - 1$  are same and given by  $b^{(2)}$  and so on. This is a simple way of reducing the number of unknown parameters from  $m$  to  $c + 1$ .

If  $\eta$  is the number of calibration-instruments, that is the number of credit derivatives used in the calibration, we set  $c = \eta - 1$ . Let  $\mathbf{a} = (a, b^{(1)}, \dots, b^{(c)})$  denote the parameters describing the model and let  $\{C_j(T; \mathbf{a})\}$  be the  $\eta$  different model spreads for the instruments used in the calibration and  $\{C_{j,M}(T)\}$  the corresponding market spreads. In  $C_j(T; \mathbf{a})$  we have emphasized that the model spreads are functions of  $\mathbf{a} = (a, b^{(1)}, \dots, b^{(c)})$  but suppressed the dependence of interest rate, payment frequency, etc. The vector  $\mathbf{a}$  is then obtained as

$$\mathbf{a} = \underset{\hat{\mathbf{a}}}{\operatorname{argmin}} \sum_{j=1}^{\eta} (C_j(T; \hat{\mathbf{a}}) - C_{j,M}(T))^2 \quad (6.1.2)$$

with the constraint that all elements in  $\mathbf{a}$  are nonnegative. Note that it would have been possible to let the jump parameters  $b_k$  be negative, as long as  $\lambda_t > 0$  for all  $t$ . In economic terms this would mean that the non-defaulted obligors benefit from the default at  $T_k$ .

The model spreads  $\{C_j(T; \mathbf{a})\}$ , such as average CDS spread  $R(T; \mathbf{a})$ , index CDS spread  $S(T; \mathbf{a})$ , CDO tranche spreads  $\{S_\gamma(T; \mathbf{a})\}$  etc. are given in closed formulas derived in the previous sections. We use Padé-approximation with scaling and squaring, (see [24]) to compute the matrix exponential, since in the present setting, it outperforms all other methods, both in computational time and accuracy. Note that this is not the case for a nonhomogeneous portfolio with a large state space  $\mathbf{E}$ , where the uniformization method is better, see [17]. The reason for the lesser performance of the uniformization method in the homogeneous CDO model is that the quantity  $\max\{|\mathbf{Q}_{j,j}| : j \in \mathbf{E}\}$  is very large, which introduces many terms in the approximation of the matrix exponential.

The initial parameters in the calibration can be rather arbitrary. The "optimal solution" for this first iteration, is taken as a the initial value in a new calibration. Repeating this procedure one, or if needed, two or three times, have in our numerical examples (see next subsections) always lead to perfect calibrations. Finding good initial parameters when using the model in practice is most likely a minor problem. This is due to the fact that calibrations are performed on a daily basis and the initial guess could simply be the optimal solution from the previous calibration.

Finally, it should be mentioned that the calibrated parameters are not likely to be unique. By perturbing the initial guesses, we have been able to get calibrations that are worse, but "close" to the optimal calibration, and where some of the parameters in the calibrated perturbed vector, are very different from the corresponding parameters in the optimal vector. We do not further pursue the discussion of potential nonuniqueness here, but rather conclude that the above phenomena is likely to occur also in other pricing models.

**6.2. Calibration to the iTraxx Europe series.** In this subsection we calibrate our model against credit derivatives on the iTraxx Europe series with maturity of five years. There are five different CDO tranche spreads with tranches  $[0, 3]$ ,  $[3, 6]$ ,  $[6, 9]$ ,  $[9, 12]$  and  $[12, 22]$ , and we also have the index CDS spreads and the average CDS spread.



First, a calibration is done against data taken from iTraxx Europe on August 4, 2004 used in e.g. [12] and [18]. Here, just as in [12] and [18], we set the average CDS spread equal to (i.e. approximated by) the index CDS spread. No market data on FtD spreads are available in this case. The iTraxx Europe spreads has changed drastically since August 2004. We therefore recalibrate our model to a more recent data set, collected at November 28<sup>th</sup>, 2006. This data also contains the average CDS spread and average FtD spread (see Table 8). All data is taken from Reuters on November 28<sup>th</sup>, 2006 and the bid, ask and mid spreads are displayed in Table 7.

In both calibrations the interest rate is set to 3%, the payment frequency is quarterly and the recovery rate is 40%.

**Table 1:** iTraxx Europe, August 4<sup>th</sup> 2004. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The [0, 3] spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
[0, 3]	27.6	27.6	0.0004514	1.635e-005
[3, 6]	168	168	0.003321	0.001977
[6, 9]	70	70.07	0.06661	0.09515
[9, 12]	43	42.91	0.09382	0.2182
[12, 22]	20	20.03	0.03304	0.1652
index	42	41.99	0.01487	0.03542
avg CDS	42	41.96	0.04411	0.105
$\Sigma$ abs.cal.err			0.2562 bp	

**Table 2:** iTraxx Europe Series 6, November 28<sup>th</sup>, 2006. The market and model spreads and the corresponding absolute errors, both in bp and in percent of the market spread. The [0, 3] spread is quoted in %. All maturities are for five years.

	Market	Model	error (bp)	error (%)
[0, 3]	14.5	14.5	0.007266	0.0005011
[3, 6]	62.5	62.41	0.08523	0.1364
[6, 9]	18	18.1	0.09727	0.5404
[9, 12]	7	6.881	0.1193	1.704
[12, 22]	3	3.398	0.3979	13.26
index	26	26.13	0.1299	0.4997
avg CDS	26.87	26.12	0.7535	2.804
$\Sigma$ abs.cal.err			1.59 bp	

We choose the partition  $\mu_1, \mu_2, \dots, \mu_6$  so that it roughly coincides with the number of defaults needed to reach the upper attachment point for each tranche, see Table 10 in

Appendix. The numerical values of the calibrated parameters  $\mathbf{a}$ , obtained via (6.1.2), are shown in Table 9 in Appendix 8.

For both data sets we also performed calibrations where some of the available market spreads were excluded from the fitting and where the model spreads for the omitted instruments were computed with the parameters obtained from the rest of the instruments in the calibration.

There were two reasons for these tests. First, we wanted to explore if the derivatives not used in the calibration, but computed with the parameters obtained from the rest of the instruments, produces model spreads that are close to the corresponding market spreads. Secondly, we wished to investigate the "robustness" of the model, that is, would the model spreads change drastically if we used different calibration instruments. For the August 4<sup>th</sup> 2004 data set, this was done for two cases. In the first fitting we excluded the index CDS and in the second, the average CDS spread was omitted in the calibration. The sum of the absolute calibration error for the two cases (and sum of total absolute model error, equal to the total calibration error and sum of absolute differences between model and market spreads for instruments not used in the calibration) were approximately 1.14 bp (1.577 bp) and 1.129 bp (1.623 bp) respectively. We can therefore, in all three calibrations, speak of a perfect fit for  $T = 5$  years. A superior fit was in this case obtained when both the average CDS spread and index CDS were included, see Table 1

We also performed the same procedure for the November 28<sup>th</sup>, 2006 data set, but now with one more case since we had one more market observation, the average FtD spread. The sum of the absolute calibration error for the three cases (and sum of total absolute model error) were approximately 1.661 bp (4.096 bp), 0.7527 bp (3.921 bp) and 3.919 bp (3.919 bp), where the last case included the average FtD-spread. Hence, once again, we can in all four cases speak of a perfect fit when  $T = 5$ . In the 2006-11-28 study we observed that the FtD model spread was very robust, that is, the computed model spreads differed very little after each calibration. This may indicate that the average FtD spread is difficult to calibrate using the model in (5.1.1). To summarize, in both data sets, the best calibrations were obtained when both the index CDS spread and average CDS spread were included, but where the average FtD-spread was excluded, see Tables 1 and 2.

Finally, since the calibrations were performed on two data sets where the corresponding spreads differed substantially, the above observations lend some confidence in the robustness of our model.

**6.3. Pricing tranchelets in a homogeneous model.** As discussed above, a tranchelet is a nonstandard CDO tranche with smaller loss-intervals than standardized tranches, see e.g. [5] or [20]. Tranchelets are typically computed for losses on  $[0, 1], [1, 2], \dots, [5, 6]$ . Currently, there are no liquid market for these instruments, so they can still be regarded as somewhat "exotic". Nevertheless, tranchelets have recently become popular and pricing these instruments are done in the same ways as for standard tranches.

In this subsection we compute the five year tranchelet spreads for  $[0, 1], \dots, [11, 12]$ , on iTraxx Europe Series 6, November 28<sup>th</sup> 2006, and iTraxx Europe, August 4<sup>th</sup>, 2004 as well as the corresponding absolute difference in % of the 2004-08-04 spreads. The computations

**Table 3:** Tranchelet spreads on iTraxx Europe, November 28<sup>th</sup> 2006 (Series 6) and August 4<sup>th</sup> 2004 and the absolute difference in % of the 2004-08-04 spreads. The [0, 1] and [1, 2] spreads are the upfront premiums on the tranche nominals, quoted in % where the running fee is 500 bp. Tranchelets above [1, 2] are expressed in bp. All maturities are five years.

Tranchelet	04/08/04	06/11/28	diff. (in %)
[0, 1]	60.85	47.93	21.25
[1, 2]	22.43	7.006	68.76
[2, 3]	488.9	245.5	49.79
[3, 4]	240.9	97.85	59.39
[4, 5]	154	54.49	64.61
[5, 6]	110.2	35.13	68.12
[6, 7]	84.29	24.26	71.22
[7, 8]	68.41	17.35	74.65
[8, 9]	57.53	12.69	77.94
[9, 10]	49.29	9.315	81.1
[10, 11]	42.53	6.676	84.3
[11, 12]	36.9	4.652	87.39

are done with parameters obtained from the calibrations in the Tables 1 and 2, where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables. The [0, 1] and [1, 2] spreads are computed with Equation (4.3) where  $\ell^{(1)}$  is replaced by a corresponding column vector adapted for [0, 1], and [1, 2] respectively, given as in Lemma 5.2. Furthermore, in (4.3),  $k_1$  is set to 0.01 for both tranchelets [0, 1] and [1, 2]. Tranchelets above [1, 2] are computed with Equation (4.2). It is interesting to note that the average for the three tranchelets between 3 and 6 are 168.4 (2004-08-04) and 62.49 (2006-11-28) which both are close to the corresponding [3, 6] spreads. The same

**Table 4:** The market spreads (used for calibration) on iTraxx Europe, November 28<sup>th</sup> 2006 (Series 6) and August 4<sup>th</sup> 2004 and the absolute difference in % of the 2004-08-04 spreads. All maturities are five years.

	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]	index	avg CDS
04/08/04	27.6	168	70	43	20	42	42
06/11/28	14.5	62.5	18	7	3	26	26.87
diff. (%)	47.46	62.8	74.29	83.72	85	38.1	36.02

holds for the averages of tranchelets between 6 to 9 and 9 to 12, which are 70.08, 18.1 and 42.91, 6.881 respectively. These observations explain why the average of the differences for the three tranchelets between 3 to 6, 6 to 9 and 9 to 12, given by 64 %, 74.6 % and 84.3

%, are close to the corresponding differences in the [3, 6], [6, 9] and [9, 12] tranche spreads, displayed in Table 4.

**6.4. Pricing  $k^{\text{th}}$ -to-default swaps on subportfolios in a homogeneous model.** In this subsection we price five year  $k^{\text{th}}$ -to-default spreads  $R_k^{(\mathbf{s})}$  with  $k = 1, \dots, 5$  for different subportfolios  $\mathbf{s}$ , of the main portfolio. The subportfolios have sizes  $|\mathbf{s}| = 5, 10, 15, 25, 30$  and the computations are done for the two different data sets, iTraxx Europe Series 6, November 28<sup>th</sup>, 2006 and iTraxx Europe August 4<sup>th</sup>, 2004. The computations are done with parameters obtained from the calibrations in the Tables 1 and 2, where all other quantities such as recovery rate, interest rate, payment frequency etc. are the same as in these tables.

**Table 5:** The five year  $k^{\text{th}}$ -to-default spreads  $R_k^{(\mathbf{s})}$  with  $k = 1, \dots, 5$  for different subportfolios  $\mathbf{s}$  in the main portfolio calibrated to iTraxx Europe, November 28<sup>th</sup> 2006 (Series 6) and August 4<sup>th</sup> 2004 and the absolute difference in % of the 2004-08-04 spreads. We consider  $|\mathbf{s}| = 5, 10, 15, 25, 30$ .

$ \mathbf{s} $	Date	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
5	04/08/04	180.9	25.19	7.002	3.037	1.404
	06/11/28	119	9.597	2.31	1.728	1.59
	diff. (%)	34.19	61.9	67.01	43.09	13.25
10	04/08/04	331	67.94	22.39	10.85	6.35
	06/11/28	226.8	30.6	6.183	2.6	1.937
	diff. (%)	31.47	54.96	72.39	76.03	69.49
15	04/08/04	467.4	117.1	41.91	21.13	12.9
	06/11/28	327.7	58.89	13.69	4.848	2.68
	diff. (%)	29.89	49.7	67.33	77.05	79.22
20	04/08/04	594.6	170.1	64.57	32.96	20.6
	06/11/28	423.1	91.73	24.34	8.69	4.234
	diff. (%)	28.84	46.07	62.31	73.63	79.44
25	04/08/04	714.9	225.5	90.06	46.15	29
	06/11/28	514.1	127.6	37.6	14	6.691
	diff. (%)	28.08	43.42	58.25	69.67	76.93

There exists liquid quoted market spreads on FtD baskets (i.e.  $k = 1$ ) and often the FtD spreads are also quoted in percent of the sum of the individual spreads in the subportfolio  $\mathbf{s}$  (see Table 8 in Appendix). No market spread on FtD swaps are available for 2004-08-04 but the model FtD-spread is 180.9 bp which is around 86 % of the SoS (sum of spreads) given by  $5 \cdot 42 = 210$  bp. As seen in Table 8 in Appendix, this is a very realistic FtD spread in terms of the SoS. Furthermore, for 2006-11-28 we have access to the average FtD market-spread which is 116.8 bp, see Table 8.

From Table 5 we see that, for fixed  $\mathbf{s}$  and  $k$ , the spreads differ substantially between the two dates. Given the difference between the market spreads in the calibration (Table 4), this should not come as a surprise. For example, when  $|\mathbf{s}| = 5$ ,  $k = 1$  the difference is 34 %, and for  $|\mathbf{s}| = 15$ ,  $k = 5$  the 2006-11-28 spread is 79 % lower than the 2004-08-04 spread. The spreads increase as the size of the portfolio increases, as they should.

For the 2006-11-28 case, the increase from a portfolio of size 5 to one of size 25 is 432% for a 1<sup>st</sup>-to-default swap, 1330% for a 2<sup>nd</sup>-to-default swap, 1628% for a 3<sup>rd</sup>-to-default swap, and for a 5<sup>th</sup>-to-default swap the increase is 421%. Further, for a portfolio of size 10 the price of a 1<sup>st</sup>-to-default swap is about 117 times higher than for a 5<sup>th</sup>-to-default swap and the corresponding ratio for a portfolio of size 15 is about 122. These ratios are much smaller than for a "isolated" portfolio, which only undergo default contagion from obligors within the basket, see [17]. Qualitatively the above results are completely as expected, however, given market spreads on CDO tranches, index CDS spreads etc. it would seem rather impossible to guess the sizes of the effects without computation.

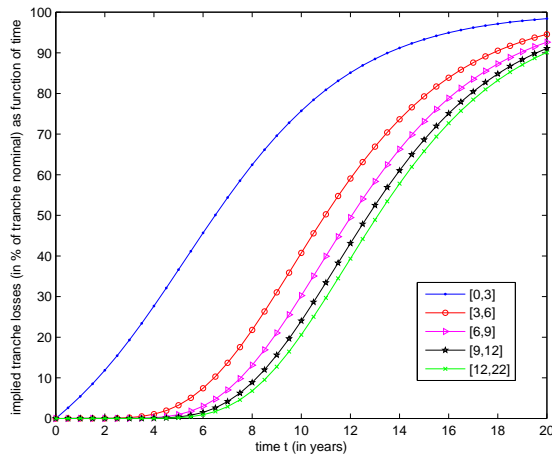
### 6.5. The implied tranche losses and implied loss in a homogeneous portfolios.

In the credit literature today, expected risk-neutral tranche losses are often called *implied* tranche losses. Here "implied" is refereing to the fact that the quantities are retrieved from market data via a model. Similarly, the implied portfolio loss refers to the expected risk-neutral portfolio loss. In this subsection we compute the expected risk-neutral portfolio loss and the implied expected tranche losses at different time points.

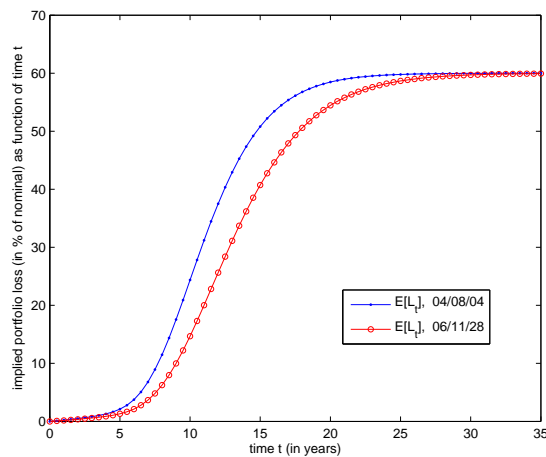
**Table 6:** The implied tranche losses in % of tranche nominal, at  $t = 3, 5, 7, 10$  for the calibrated CDO portfolios on iTraxx Europe Series 6, November 28<sup>th</sup> 2006, and iTraxx Europe, August 4<sup>th</sup>, 2004 and the absolute differences in % of the 2004-08-04 tranche losses.

$t$	Date	[0, 3]	[3, 6]	[6, 9]	[9, 12]	[12, 22]
3	04/08/04	26.52	0.7142	0.1014	0.03198	0.005744
	06/11/28	19.31	0.2082	0.01647	0.002157	0.0004121
	diff. (%)	27.18	70.85	83.75	93.26	92.83
5	04/08/04	49.26	8.649	3.67	2.258	1.059
	06/11/28	36.61	3.255	0.954	0.3641	0.1802
	diff. (%)	25.67	62.37	74.01	83.88	82.99
7	04/08/04	69.28	28.61	18.7	14.74	10.13
	06/11/28	54.39	13.7	7.005	4.161	2.9
	diff. (%)	21.49	52.09	62.54	71.78	71.38
10	04/08/04	87.91	63.57	54.27	49.67	43.12
	06/11/28	75.73	40.75	30.24	24.01	20.58
	diff. (%)	13.86	35.89	44.27	51.65	52.28

These are important quantities for a credit manager and Lemma 4.1 and Lemma 5.2 provides formulas for computing them. We study  $100 \cdot \mathbb{E} \left[ L_t^{(\gamma)} \right] / \Delta k_\gamma$  for 3, 5, 7 and 10



**Figure 1:** The implied tranche losses in % of tranche nominal for the 2006-11-28 portfolio.

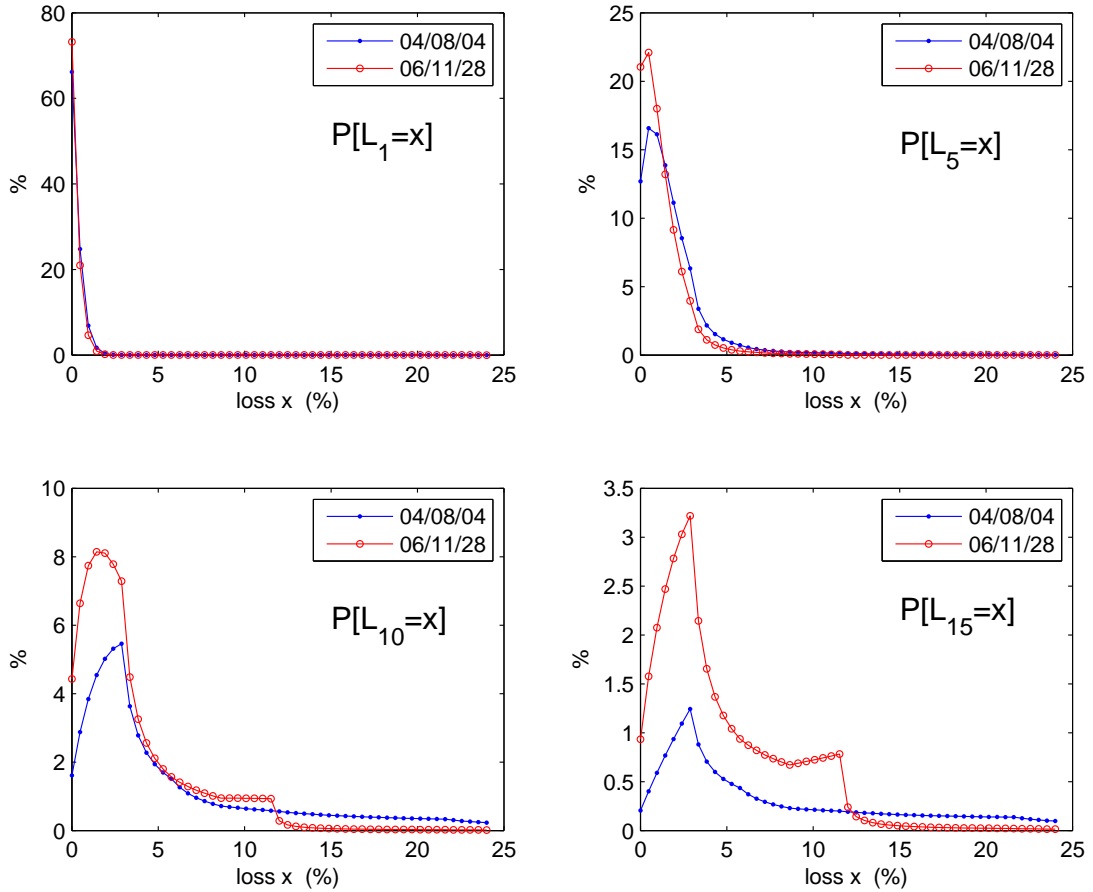


**Figure 2:** The implied portfolio losses in % of nominal, for the 2004-08-04 and 2006-11-28 portfolios.

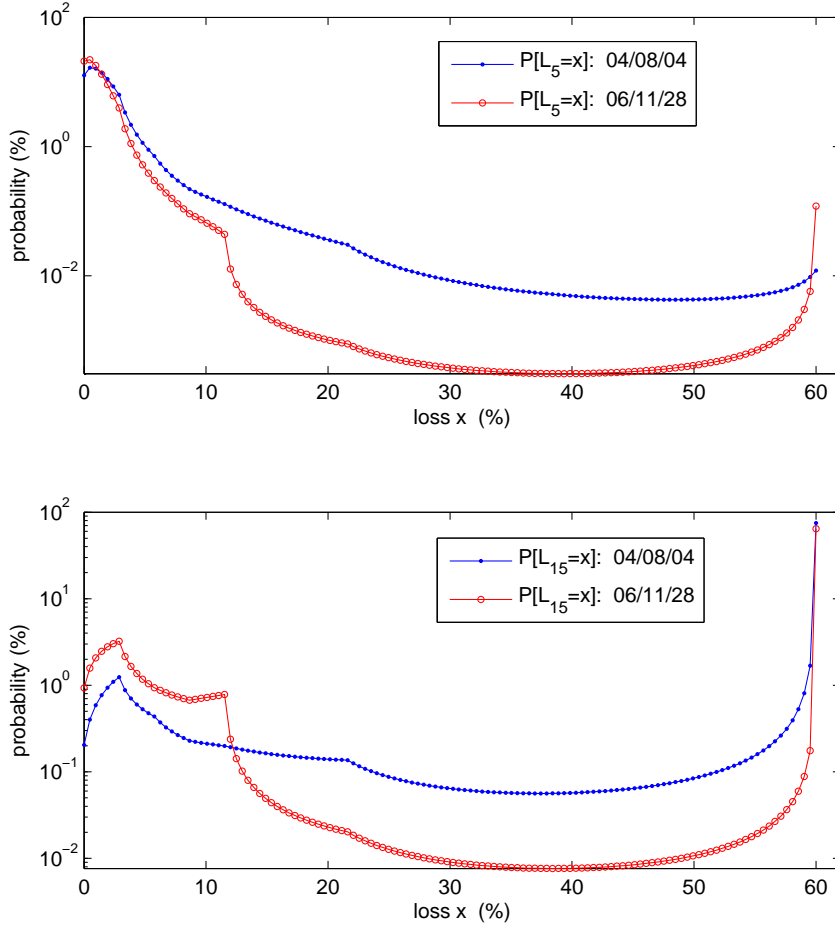
years on CDO portfolios calibrated against iTraxx Europe Series 6, November 28<sup>th</sup> 2006, and iTraxx Europe, August 4<sup>th</sup>, 2004. Just as for previous computations, the corresponding tranche losses differ substantially between the two dates. For example, in the 2006-11-28 case, the tranche loss on  $[0, 3]$  for  $t = 3$  is 27 % smaller than the corresponding quantity for the 2004-08-04 collection, but this differences drastically increases for the upper tranches,  $[6, 9]$ ,  $[9, 12]$  to 84% and 93%, as seen in Table 6. Further, for the 2006-11-28 case, we clearly

see the effect of default contagion on the upper tranche losses, making them lie close to each other, see Figure 1. From Figure 2 we conclude that our model, with a constant recovery rate of 40%, calibrated to market spreads on the five year iTraxx Europe Series implies that the whole portfolio has defaulted within approximately 30 years (for both data sets). In reality, this will likely not happen, since risk-neutral (implied) default probabilities are substantially larger than the "real", so called actuarial, default probabilities.

**6.6. The implied loss distribution in a homogeneous portfolio.** In this subsection we study the implied distribution for the loss process  $L_t$  at different time points. Since we are considering constant recovery rates, then for every  $t$ , the distribution of  $L_t$  is discrete and formally the values for  $\mathbb{P}[L_t = x]$  should be displayed as bars at  $x = k(1 - \phi)/m$  where  $0 \leq k \leq m$ .



**Figure 3:** The implied loss distributions for the 2004-08-04 and 2006-11-28 portfolios.



**Figure 4:** The implied loss distributions (in log-scale) for the 2004-08-04 and 2006-11-28 portfolios.

However, since there are totally 126 different outcomes we do not bother about this and connect the graph continuously between each discrete probability. The loss probabilities are computed by using that  $L_t = L(Y_t)$  so  $\mathbb{P}[L_t = k(1 - \phi)/m] = \mathbb{P}[Y_t = k] = \alpha e^{Q_t} e_k$  for  $k = 0, 1, \dots, m$ , see Corollary 5.1.

In Figure 3 for  $0 < x < 12$ , the implied loss probabilities in the 2006-11-28 case are bigger than their 2004-04-28 counterparts, at several occasions in time  $t$ , which at first glance may contradict the results in Table 6. However, a more careful study, using a log-scale, shows that for  $20 < x < 50$  and at most time points  $t$ , the 2004-04-28 loss distribution is about 10 times bigger than the corresponding values for the 2006-11-28 case, see Figure 4. This



supports the results in Table 6 where the expected tranche losses for the 2004-04-28 case are always bigger than in the 2006-11-28 case.

## 7. CONCLUSIONS

In this paper we have derived closed-form expressions for CDO tranche spreads and index CDS spreads. This is done in a inhomogeneous model where dynamic default dependencies among obligors are expressed in an intuitive, direct and compact way. By specializing this model to a homogenous portfolio, we show that the CDO and index CDS formulas simplify considerably in a symmetric model. The same method are used to derive  $k^{\text{th}}$ -to-default swap spreads for subportfolios in the main CDO portfolio. In this setting, we calibrate a symmetric portfolio against credit derivatives on the iTraxx Europe series for a fixed maturity of five years. We do this at two different dates, where the corresponding market spreads differ substantially. In both cases we obtain perfect fits. These two calibrations therefore lends some confidence to the robustness of our model.

In the calibrated portfolios, we compute tranchelet spreads and investigate  $k^{\text{th}}$ -to-default swap spreads as function of the portfolio size. Further, the implied tranche losses and the implied loss distributions are also extracted. All these computations and investigations would be difficult to perform without having convenient formulas for the quantities that we want to study. Furthermore, given the recovery rate, the number of model parameters are as many as the market instruments used in the calibration. This implies that all calibrations are performed without inserting "fictitious" numerical values for some of the parameters, making the calibration more realistic.

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## 8. APPENDIX

Table 7 shows the market spreads collected from iTraxx Europe Series 6, November 28<sup>th</sup>, 2006 and taken from Reuters. Table 8 shows the FtD spreads, i.e. 1<sup>st</sup>-to-defaults spreads for 6 standardized subportfolios on iTraxx Europe Series 6, launched September 20<sup>th</sup>, 2006. Each basket consist of five obligors that are taken from a sector in the iTraxx Series 6 (Autos, Energy, Industrial, TMT, Consumers and Financial). The names of the obligors in each basket as well as the selection criteria can be found on the webpage for iboxx. In the financial FtD basket, we have used the subordinated FtD spread, since the senior spread is much smaller (30 bp) than the other spreads, which will pull down the average mid FtD spread to 112.25 bp.

The numerical values of the calibrated parameters  $\mathbf{a}$ , obtained via (6.1.2), are shown in Table 9 and the partition (see Equation (6.1.1)) in Table 10.

**Table 7:** The market bid, ask and mid spreads for iTraxx Europe (Series 6), November 28<sup>th</sup>, 2006. All data is taken from Reuters. The mid spreads, i.e. average of the bid and ask spread, are used in the calibration in Section 6.

	bid	ask	mid	time
[0, 3]		14.5	14.5	28 Nov, 18:23
[3, 6]	60	65	62.5	28 Nov, 17:14
[6, 9]	16.5	19.5	18	28 Nov, 13:36
[9, 12]	5.5	8.5	7	28 Nov, 13:36
[12, 22]	2	4	3	28 Nov, 13:36
index	25.75	26.25	26	28 Nov, 18:34
avg CDS	25.94	27.8	26.87	28 Nov, 19:40

**Table 8:** The market bid, ask and mid spreads for different FtD spreads on subsectors of iTraxx Europe (Series 6), November 28<sup>th</sup>, 2006. Each subportfolio have five obligors. We also display the sum of CDS-spreads (SoS) in each basket, as well as the mid FtD spreads in % of SoS. The mid spread is used in the calibration in Section 6.

Sector	bid	ask	mid	SoS	mid/SoS %	time
Autos	154	166	160	202	79.21 %	28 Nov, 10:26
Energy	65	71	68	86	79.07 %	28 Nov, 10:26
Industrial	114	123	118.5	141	84.04 %	28 Nov, 10:26
TMT	167	188	177.5	217	81.8 %	28 Nov, 10:26
Consumers	113	122	117.5	140	83.93 %	28 Nov, 10:26
Financial	55	63	59	79	74.68 %	28 Nov, 10:26
average	111.3	122.2	116.8	144.2	80.98 %	

**Table 9:** The calibrated parameters that gives the model spreads in the Tables 1 and 2.

	$a$	$b^{(1)}$	$b^{(2)}$	$b^{(3)}$	$b^{(4)}$	$b^{(5)}$	$b^{(6)}$	
04/08/04	33.0	16.4	84.5	145	86.4	124	514	$\times 10^{-4}$
06/11/28	24.9	13.9	73.6	62.4	0.823	2162	4952	$\times 10^{-4}$

**Table 10:** The integers  $1, \mu_1, \mu_2, \dots, \mu_c$  are partitions of  $\{1, 2, \dots, m\}$  used in the models that generates the spreads in the Tables 1 and 2

partition	$\mu_1$	$\mu_2$	$\mu_3$	$\mu_4$	$\mu_5$	$\mu_6$
	7	13	19	25	46	125

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