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MODELLING DEFAULT CONTAGION USING MULTIVARIATE PHASE-TYPE DISTRIBUTIONS

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Abstract. We model dynamic credit portfolio dependence by using default contagion in an intensity-based framework. Two different portfolios (with 10 obligors), one in the European auto sector, the other in the European financial sector, are calibrated against their market CDS spreads and the corresponding CDS-correlations. After the calibration, which are perfect for the banking portfolio, and good for the auto case, we study several quantities of importance in active credit portfolio management. For example, implied multivariate default and survival distributions, multivariate conditional survival distributions, implied default correlations, expected default times and expected ordered defaults times. The default contagion is modelled by letting individual intensities jump when other defaults occur, but be constant between defaults. This model is translated into a Markov jump process, a so called multivariate phase-type distribution, which represents the default status in the credit portfolio. Matrix-analytic methods are then used to derive expressions for the quantities studied in the calibrated portfolios.

1. Introduction

In recent years, understanding and modelling default dependency has attracted much interest. A main reason is the incentive to optimize regulatory capital in credit portfolios, provided by new regulatory rules such as Basel II. Another reason is the growing financial market of products whose payoffs are contingent on the default behavior of a whole credit portfolio consisting of, for example, mortgage loans, corporate bonds or single-name credit default swaps (CDS-s).

In this paper we model dynamic credit portfolio dependence by using default contagion and consider two different portfolios, one in the European auto sector, the other in the European financial sector. Both baskets consist of 10 companies which are calibrated against their market CDS spreads and the corresponding CDS correlations, resulting in a

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perfect fit for the banking case and good fit for the auto case. We then study the implied joint default and survival distributions, the implied univariate and bivariate conditional survival distributions, the implied default correlations, and the implied expected default times and expected ordered defaults times. These quantities are of importance in active credit portfolio management.

We us an intensity based model where default dependencies among obligors are expressed in an intuitive and compact way. The financial interpretation is that the individual default intensities are constant, except at the times when other defaults occur: then the default intensity for each obligor jumps by an amount representing the influence of the defaulted entity on that obligor. This model is then translated into a Markov jump process, which leads to so called multivariate phase-type distributions, first introduced in [3]. This translation makes it possible to use a matrix-analytic approach to derive practical formulas for all quantities that we want to study. The contribution of this paper is to adapt results from [3] to credit portfolio applications. Special attention is given how to retrieve the model parameters from market CDS spreads and their CDS-correlations.

The framework used here is the same as in [19], where the authors consider CDS and k^{th}-to default spreads and in [18] where the same technique is applied to synthetic CDO tranches and index CDS-s. In this paper however, we focus on multivariate default and survival distributions. As mentioned above, computing such quantities is at the core of active credit portfolio management. The paper is an extension of Chapter 6 in the licentiate thesis [17]. Default contagion in an intensity based setting have previously also been studied in for example [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [21], [22],[23], [24], [25], [26] and [28]. The material in all these papers and books are related to the results discussed here.

The rest of this paper is organized as follows. Section 2 contains the formal definition of default contagion used in this paper, given in terms of default intensities. It is then used to construct such default times as hitting times of a Markov jump process. The joint distribution of these hitting times is called a multivariate phase-type distribution, see [3]. The results in Section 3 give convenient analytical formulas for multivariate default and survival distributions, conditional multivariate distributions, marginal default distributions, multivariate default densities, default correlations, and expected default times. These are the main theoretical contribution of this paper. Some of the results in this section have previously been stated in [3], but without proofs. Section 4 gives formulas for CDS-spreads. They are our main calibration instruments. We provide a detailed description of the calibration against CDS spreads and their correlations. Special attention is given to the relation between market CDS-correlations and the corresponding default correlations. Furthermore, we discuss how to deal with negative jumps in the intensities, which are required if there are negative CDS-correlations. In Section 5 we use the results of Section 3 for our numerical investigations. Two CDS portfolios are calibrated against market CDS spreads and their CDS-correlations. We then study several quantities of interest in credit portfolio management. Section 6 is devoted to numerical issues and the final section, Section 7, summarizes and discusses the results.

2. Intensity based models reinterpreted as Markov jump processes: multivariate phase-type distributions

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then reinterpreted in terms of a Markov jump process, a so called multivariate phase-type distribution, introduced in [3]. Such constructions have largely been developed for queueing theory and reliability applications, see e.g. [1] and [3]).

For the default times $\tau_1, \tau_2 \dots, \tau_m$, define the point process $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we will for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$. The model studied in this paper is specified by requiring that the default intensities have the following form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad t \leq \tau_i, \tag{2.1}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (2.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

Equation (2.1) determines the default times through their intensities as well as their joint distribution. However, it is by no means obvious how to find these expressions. Here we will use the following observation, proved in [19].

Proposition 2.1. *There exists a Markov jump process $(Y_t)_{t \geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\left\{t > 0 : Y_t \in \Delta_i\right\}, \qquad i = 1, 2, \dots, m, \tag{2.2}$$

have intensities (2.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2, \dots, \tau_m)$ can be obtained from $\{Y_t\}_{t \geq 0}$.

The joint distribution of $(\tau_1, \tau_2, \dots, \tau_m)$ is sometimes called a multivariate phase-type distribution (MPH), and was first introduced in [3]. In this paper, Proposition 2.1 is throughout used for computing distributions. However, we still use Equation (2.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive.

Each state $\boldsymbol{j}$ in $\boldsymbol{E}$ is of the form $\boldsymbol{j} = \{j_1, \dots j_k\}$ which is a subsequence of $\{1, \dots m\}$ consisting of k integers, where $1 \leq k \leq m$. The interpretation is that on $\{j_1, \dots j_k\}$ the obligors in the set have defaulted. Before we continue, further notation are needed. In the sequel, we let $\boldsymbol{Q}$ and $\boldsymbol{\alpha}$ denote the generator and initial distribution on $\boldsymbol{E}$ for the Markov jump process in Proposition 2.1. The generator $\boldsymbol{Q}$ is found by using the structure of $\boldsymbol{E}$, the definition of the states $\boldsymbol{j}$, and Equation (2.1). The states are ordered so that $\boldsymbol{Q}$ is upper triangular, see [19]. In particular, the final state $\{1,\dots m\}$ is absorbing and $\{0\}$ is always the starting state. The latter implies that $\boldsymbol{\alpha} = (1, 0, \dots, 0)$. Furthermore, define the probability vector $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j}\in\boldsymbol{E}}$. From Markov theory we know that

$$\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}, \quad \text{and} \quad \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_{\boldsymbol{j}}, \tag{2.3}$$

where $\boldsymbol{e}_{\boldsymbol{j}} \in \mathbb{R}^{|\boldsymbol{E}|}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Recall that $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

3. Using Multivariate Phase-type distributions and the matrix-analytic approach to find multivariate default distributions

In this section we derive expressions for various quantities of importance in active credit portfolio management. The portfolio consists of m obligors with default intensities (2.1). Subsection 3.1 presents formulas for multivariate default and survival distributions, conditional multivariate default distributions, and multivariate default densities. In subsection 3.2 we briefly restate some expressions for marginal survival distributions, originally presented in [19]. These distributions are needed in Section 4. Analytical formulas for the default correlations are given in Subsection 3.3. Finally, in Subsection 3.4 we present compact expressions for the moments of the default times and the ordered default times.

3.1. **The multivariate default distributions.** In this subsection we derive formulas for multivariate default and survival distributions, conditional multivariate default distributions, and multivariate default densities. Let $\boldsymbol{G}_i$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i^C\}} \quad \text{and} \quad (\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.1}$$

Further, for a vector $(t_1, t_2, \dots, t_m)$ in $\mathbb{R}_+^m = [0,\infty)^m$, let the ordering of $(t_1, t_2, \dots, t_m)$ be $t_{i_1} < t_{i_2} < \dots < t_{i_m}$ where $(i_1, i_2, \dots, i_m)$ is a permutation of $(1, 2, \dots, m)$. The following proposition was stated in [3], but without a proof.

Proposition 3.1. *Consider m obligors with default intensities* (2.1). *Let $(t_1, t_2, \dots, t_m) \in \mathbb{R}_+^m$ and let $t_{i_1} < t_{i_2} < \dots < t_{i_m}$ be its ordering. Then,*

$$\mathbb{P}[\tau_1 > t_1, \dots, \tau_m > t_m] = \boldsymbol{\alpha} \left(\prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \boldsymbol{1} \tag{3.1.2}$$

where $t_{i_0} = 0$.

Proof. First, note that

$$\begin{aligned}\mathbb{P}\left[\tau_1 > t_1, \dots, \tau_m > t_m\right] &= \mathbb{P}\left[\tau_{i_1} > t_{i_1}, \dots, \tau_{i_m} > t_{i_m}\right] \\ &= \mathbb{P}\left[Y_{t_{i_1}} \in \Delta^C_{i_1}, \dots, Y_{t_{i_m}} \in \Delta^C_{i_m}\right] \\ &= \mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} \in \Delta^C_{i_1}, \dots, Y_{t_{i_m}} \in \Delta^C_{i_m}\right] \\ &= \sum_{\boldsymbol{j}_{i_1} \in \Delta^C_{i_1}} \cdots \sum_{\boldsymbol{j}_{i_m} \in \Delta^C_{i_m}} \mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \dots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right]\end{aligned} \tag{3.1.3}$$

where $\mathbf{0} = \{0\}$ is the state representing that no default have occurred. Further,

$$\begin{aligned}&\mathbb{P}\left[Y_0 = \mathbf{0}, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \dots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right] \\ &= \mathbb{P}\left[Y_0 = \mathbf{0}\right] \mathbb{P}\left[Y_{t_{i_1}} = \boldsymbol{j}_{i_1} \,\middle|\, Y_0 = \mathbf{0}\right] \cdot \ldots \cdot \mathbb{P}\left[Y_{t_{i_m}} = \boldsymbol{j}_{i_m} \,\middle|\, Y_{t_{m-1}} = \boldsymbol{j}_{i_{m-1}}\right] \\ &= \boldsymbol{\alpha} e^{\boldsymbol{Q} t_{i_1}} \boldsymbol{e}_{\boldsymbol{j}_{i_1}} \boldsymbol{e}^T_{\boldsymbol{j}_{i_1}} e^{\boldsymbol{Q}\left(t_{i_2} - t_{i_1}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_2}} \boldsymbol{e}^T_{\boldsymbol{j}_{i_2}} \cdot \ldots \cdot \boldsymbol{e}_{\boldsymbol{j}_{i_{m-1}}} \boldsymbol{e}^T_{\boldsymbol{j}_{i_{m-1}}} e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_m}}\end{aligned} \tag{3.1.4}$$

where the first equality follows from the Markov property of Y_t, and $\mathbb{P}\left[Y_0 = \mathbf{0}\right] = 1$. The second equality is because

$$\mathbb{P}\left[Y_t = \boldsymbol{j}_{i_k} \,\middle|\, Y_s = \boldsymbol{j}_{i_{k-1}}\right] = \mathbb{P}\left[Y_{t-s} = \boldsymbol{j}_{i_k} \,\middle|\, Y_0 = \boldsymbol{j}_{i_{k-1}}\right] = \left(\boldsymbol{e}^T_{\boldsymbol{j}_{i_{k-1}}} e^{\boldsymbol{Q}(t-s)}\right)_{\boldsymbol{j}_{i_k}}$$

since Y_t is a homogeneous Markov process. Next,

$$\sum_{\boldsymbol{j}_{i_k} \in \Delta^C_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}} \boldsymbol{e}^T_{\boldsymbol{j}_{i_k}} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} = \left(\sum_{\boldsymbol{j}_{i_k} \in \Delta^C_{i_k}} \boldsymbol{e}_{\boldsymbol{j}_{i_k}} \boldsymbol{e}^T_{\boldsymbol{j}_{i_k}}\right) e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} = \boldsymbol{G}_{i_k} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \tag{3.1.5}$$

for $k = 1, 2, \dots m - 1$, and

$$\sum_{\boldsymbol{j}_{i_m} \in \Delta^C_{i_m}} e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{e}_{\boldsymbol{j}_{i_m}} = e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{G}_{i_m} \mathbf{1}. \tag{3.1.6}$$

Hence, inserting the equations (3.1.4)-(3.1.6) into (3.1.3) shows that (3.1.2) hold. □

Let $(t_{i_1}, t_{i_2}, \dots, t_{i_m})$ be the ordering of $(t_1, t_2, \dots, t_m) \in \mathbb{R}^m_+$ and fix a p, $1 \leq p \leq m - 1$. We next consider conditional distributions of the types

$$\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \dots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \dots, \tau_{i_p} \leq t_{i_p}\right]$$

and

$$\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \dots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \dots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right]$$

There is a subtle but important difference between these two probabilities. The conditioning in the first expression includes the possibility that all obligors have defaulted before t_{i_p}. This is not the case in the second one, where the event excludes the possibility that other obligors than $i_1, \dots, i_p$ default before t_{i_p}. These probabilities may of course be computed from (3.1.2) without any further use of the structure of the problem. However, using this structure leads to compact formulas. For this, further notation is needed. Define Δ as the final absorbing state for Y_t, i.e.

$$\Delta = \bigcap_{i=1}^{m} \Delta_i, \tag{3.1.7}$$

and let $\boldsymbol{F}_i$ and $\boldsymbol{H}_i$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$\left(\boldsymbol{F}_i\right)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i\setminus\Delta\}} \quad \text{and} \quad \left(\boldsymbol{F}_i\right)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.8}$$

$$\left(\boldsymbol{H}_i\right)_{\boldsymbol{j},\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i\}} \quad \text{and} \quad \left(\boldsymbol{H}_i\right)_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j} \neq \boldsymbol{j}'. \tag{3.1.9}$$

The following proposition is useful.

Proposition 3.2. *Consider m obligors with default intensities* (2.1)*. Let* $(t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ *and let* $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ *be its ordering. If* $1 \leq p \leq m-1$ *then,*

$$\begin{aligned}&\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right] \\ &= \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1}.\end{aligned} \tag{3.1.10}$$

Further,

$$\begin{aligned}&\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}\right] \\ &= \frac{\boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1}}{\boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{H}_{i_k} \right) \mathbf{1}}.\end{aligned} \tag{3.1.11}$$

and

$$\begin{aligned}&\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, T_{p+1} > t_{i_p}\right] \\ &= \frac{\boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1}}{\boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \mathbf{1}}\end{aligned} \tag{3.1.12}$$

where $t_{i_0} = 0$*.*

Proof. First we prove (3.1.10). Similarly as in the proof of Proposition 3.1

$$\begin{aligned}&\mathbb{P}\left[\tau_{i_1} \leq t_{i_1}, \ldots, \tau_{i_p} \leq t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right] \\ &= \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1} \setminus \Delta, \ldots, Y_{t_{i_p}} \in \Delta_{i_p} \setminus \Delta, Y_{t_{i_{p+1}}} \in \Delta_{i_{p+1}}^C, \ldots, Y_{t_{i_m}} \in \Delta_{i_m}^C\right] \\ &= \sum_{\boldsymbol{j_0}\in\boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1}\in\Delta_{i_1}\setminus\Delta} \cdots \sum_{\boldsymbol{j}_{i_p}\in\Delta_{i_p}\setminus\Delta} \sum_{\boldsymbol{j}_{i_{p+1}}\in\Delta_{i_{p+1}}^C} \cdots \sum_{\boldsymbol{j}_{i_m}\in\Delta_{i_m}^C} \mathbb{P}\left[Y_0 = \boldsymbol{j}_0, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_m}} = \boldsymbol{j}_{i_m}\right] \\ &= \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \left(\prod_{k=p+1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1}.\end{aligned}$$

Here the last equality follows from similar arguments as in the equations (3.1.4)-(3.1.6) in Proposition 3.1, using the definition of the matrix $\boldsymbol{F}_k$.

To prove (3.1.11) it is enough to show that

$$\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}\right] = \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{H}_{i_k} \right) \mathbf{1}$$

since Equation (3.1.11) then follows from (3.1.10) and the definition of conditional probabilities. Now,

$$\begin{aligned}
\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}\right] &= \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1}, \ldots, Y_{t_{i_p}} \in \Delta_{i_p}\right] \\
&= \sum_{\boldsymbol{j}_0 \in \boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1}} \cdots \sum_{\boldsymbol{j}_{i_p} \in \Delta_{i_p}} \mathbb{P}\left[Y_0 = \boldsymbol{j}_0, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_p}} = \boldsymbol{j}_{i_p}\right] \\
&= \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{H}_{i_k} \right) \mathbf{1}
\end{aligned}$$

where the last equality follows from arguments as in Proposition 3.1, using the definition of the matrix $\boldsymbol{H}_k$. Finally, for Equation (3.1.12), note that

$$\begin{aligned}
&\mathbb{P}\left[\tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m} \,\middle|\, \tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, T_{p+1} > t_{i_p}\right] \\
&= \frac{\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, \tau_{i_{p+1}} > t_{i_{p+1}}, \ldots, \tau_{i_m} > t_{i_m}\right]}{\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, T_{p+1} > t_{i_p}\right]}.
\end{aligned}$$

Hence, by using (3.1.10) it is enough to show that

$$\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, T_{p+1} > t_{i_p}\right] = \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \mathbf{1}.$$

Let $\boldsymbol{E}_n$ be the set of states representing precisely n defaults. Then,

$$\begin{aligned}
&\mathbb{P}\left[\tau_{i_1} \le t_{i_1}, \ldots, \tau_{i_p} \le t_{i_p}, T_{p+1} > t_{i_p}\right] \\
&= \mathbb{P}\left[Y_{t_{i_1}} \in \Delta_{i_1}, \ldots, Y_{t_{i_p}} \in \Delta_{i_p}, Y_{t_{i_p}} \in \bigcup_{k=p+1}^{m} \boldsymbol{E}_{i_k}\right] \\
&= \mathbb{P}\left[Y_0 \in \boldsymbol{E}, Y_{t_{i_1}} \in \Delta_{i_1} \setminus \Delta, \ldots, Y_{t_{i_p}} \in \Delta_{i_p} \setminus \Delta\right] \\
&= \sum_{\boldsymbol{j}_0 \in \boldsymbol{E}} \sum_{\boldsymbol{j}_{i_1} \in \Delta_{i_1} \setminus \Delta} \cdots \sum_{\boldsymbol{j}_{i_p} \in \Delta_{i_p} \setminus \Delta} \mathbb{P}\left[Y_0 = \boldsymbol{j}_0, Y_{t_{i_1}} = \boldsymbol{j}_{i_1}, \ldots, Y_{t_{i_p}} = \boldsymbol{j}_{i_p}\right] \\
&= \boldsymbol{\alpha} \left(\prod_{k=1}^{p} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{F}_{i_k} \right) \mathbf{1}
\end{aligned}$$

where the second equality comes from the fact that Δ is an absorbing state representing default of all obligors. The last equality follows from arguments as in Proposition 3.1, using the definition of the matrix $\boldsymbol{F}_k$. □

The following corollary is an immediate consequence of Equation (3.1.10) in Proposition 3.2.

Corollary 3.3. *Consider* m *obligors with default intensities* (2.1). *Let* $\{i_1, \dots, i_p\}$ *and* $\{j_1, \dots, j_q\}$ *be two disjoint subsequences in* $\{1, \dots, m\}$. *If* $t < s$ *then*

$$\mathbb{P}\left[\tau_{i_1} > t, \dots, \tau_{i_p} > t, \tau_{j_1} < s, \dots, \tau_{j_q} < s\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \left(\prod_{k=1}^{p} \boldsymbol{G}_{i_k} \right) e^{\boldsymbol{Q}(s-t)} \left(\prod_{k=1}^{q} \boldsymbol{H}_{j_k} \right) \mathbf{1}$$

and for $s < t$

$$\mathbb{P}\left[\tau_{i_1} > t, \dots, \tau_{i_p} > t, \tau_{j_1} < s, \dots, \tau_{j_q} < s\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}s} \left(\prod_{k=1}^{q} \boldsymbol{F}_{j_k} \right) e^{\boldsymbol{Q}(t-s)} \left(\prod_{k=1}^{p} \boldsymbol{G}_{i_k} \right) \mathbf{1}.$$

We can of course generalize, the above proposition for three time points $t < s < u$, four time points $t < s < u <$ etc. Using the notation of Corollary 3.3 we conclude that if $t < s$ then

$$\mathbb{P}\left[\tau_{j_1} < s, \dots, \tau_{j_q} < s \,\middle|\, \tau_{i_1} > t, \dots, \tau_{i_p} > t\right] = \frac{\boldsymbol{\alpha} e^{\boldsymbol{Q}t} \left(\prod_{k=1}^{p} \boldsymbol{G}_{i_k} \right) e^{\boldsymbol{Q}(s-t)} \left(\prod_{k=1}^{q} \boldsymbol{H}_{j_k} \right) \mathbf{1}}{\boldsymbol{\alpha} e^{\boldsymbol{Q}t} \left(\prod_{k=1}^{p} \boldsymbol{G}_{i_k} \right) \mathbf{1}}$$

and for $s < t$

$$\mathbb{P}\left[\tau_{i_1} > t, \dots, \tau_{i_p} > t \,\middle|\, \tau_{j_1} < s, \dots, \tau_{j_q} < s\right] = \frac{\boldsymbol{\alpha} e^{\boldsymbol{Q}s} \left(\prod_{k=1}^{q} \boldsymbol{F}_{j_k} \right) e^{\boldsymbol{Q}(t-s)} \left(\prod_{k=1}^{p} \boldsymbol{G}_{i_k} \right) \mathbf{1}}{\boldsymbol{\alpha} e^{\boldsymbol{Q}s} \left(\prod_{k=1}^{q} \boldsymbol{H}_{j_k} \right) \mathbf{1}}.$$

Our next task is to find the probability density $f(t_1, \dots, t_m)$ of the multivariate random variable $(\tau_1, \dots, \tau_m)$. For $(t_1, t_2, \dots, t_m)$, let $(t_{i_1}, t_{i_2}, \dots, t_{i_m})$ be its ordering where $(i_1, i_2, \dots, i_m)$ is a permutation of $(1, 2, \dots, m)$. We denote $(i_1, i_2, \dots, i_m)$ by $\boldsymbol{i}$, that is, $\boldsymbol{i} = (i_1, i_2, \dots, i_m)$. Furthermore, in view of the above notation, we let $f_{\boldsymbol{i}}(t_1, \dots, t_m)$ denote the restriction of $f(t_1, \dots, t_m)$ to the set $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$. The following proposition was stated in [3], but without a proof.

Proposition 3.4. *Consider* m *obligors with default intensities* (2.1). *Let* $(t_1, t_2, \dots, t_m) \in \mathbb{R}_+^m$ *and let* $t_{i_1} < t_{i_2} < \ldots < t_{i_m}$ *be its ordering. Then, with notation as above*

$$f_{\boldsymbol{i}}(t_1, \dots, t_m) = (-1)^m \boldsymbol{\alpha} \left(\prod_{k=1}^{m-1} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \left(\boldsymbol{Q}\boldsymbol{G}_{i_k} - \boldsymbol{G}_{i_k}\boldsymbol{Q} \right) \right) e^{\boldsymbol{Q}\left(t_{i_m} - t_{i_{m-1}}\right)} \boldsymbol{Q}\boldsymbol{G}_{i_m} \mathbf{1} \tag{3.1.13}$$

where $t_{i_0} = 0$.

Proof. By Proposition 3.1, since the order of partial differentiation is irrelevant

$$\begin{aligned} f_{\boldsymbol{i}}(t_1, \dots, t_m) &= (-1)^m \frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}} \mathbb{P}\left[\tau_{i_1} > t_{i_1}, \dots, \tau_{i_p} > t_{i_p}\right] \\ &= (-1)^m \boldsymbol{\alpha} \left(\frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}} \prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k} - t_{i_{k-1}}\right)} \boldsymbol{G}_{i_k} \right) \mathbf{1} \end{aligned} \tag{3.1.14}$$

where $t_{i_0} = 0$. First, note that

$$
\begin{aligned}
\frac{\partial}{\partial t_{i_1}} \prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} &= \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\boldsymbol{Q}\boldsymbol{G}_{i_1} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} \\
&\quad - \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\boldsymbol{G}_{i_1} e^{\boldsymbol{Q}\left(t_{i_2}-t_{i_1}\right)}\boldsymbol{Q}\boldsymbol{G}_{i_2} \prod_{k=3}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} \\
&= \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\boldsymbol{Q}\boldsymbol{G}_{i_1} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} \qquad (3.1.15) \\
&\quad - \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\boldsymbol{G}_{i_1}\boldsymbol{Q} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} \\
&= \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\left(\boldsymbol{Q}\boldsymbol{G}_{i_1} - \boldsymbol{G}_{i_1}\boldsymbol{Q}\right) \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k}
\end{aligned}
$$

where the second equality is due to the fact that $\boldsymbol{e}^{\boldsymbol{Q}t}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{e}^{\boldsymbol{Q}t}$. Next, (3.1.15) implies that

$$
\frac{\partial^2}{\partial t_{i_1}\partial t_{i_2}} \prod_{k=1}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k} = \boldsymbol{e}^{\boldsymbol{Q}t_{i_1}}\left(\boldsymbol{Q}\boldsymbol{G}_{i_1} - \boldsymbol{G}_{i_1}\boldsymbol{Q}\right) \frac{\partial}{\partial t_{i_2}} \prod_{k=2}^{m} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)}\boldsymbol{G}_{i_k}. \qquad (3.1.16)
$$

The derivative of the product in the right-hand side in Equation (3.1.16) is treated exactly as in (3.1.15) but now with t_{i_2} instead of t_{i_1}. Hence, by repeating this procedure for $k = 3, \ldots, m-1$ and noting that

$$
\frac{\partial}{\partial t_{i_m}} e^{\boldsymbol{Q}\left(t_{i_m}-t_{i_{m-1}}\right)}\boldsymbol{G}_{i_m} = e^{\boldsymbol{Q}\left(t_{i_m}-t_{i_{m-1}}\right)}\boldsymbol{Q}\boldsymbol{G}_{i_m}
$$

and inserting the results in Equation (3.1.14) finally yields

$$
f_{\boldsymbol{i}}(t_1, \ldots, t_m) = (-1)^m \boldsymbol{\alpha} \left(\prod_{k=1}^{m-1} e^{\boldsymbol{Q}\left(t_{i_k}-t_{i_{k-1}}\right)} \left(\boldsymbol{Q}\boldsymbol{G}_{i_k} - \boldsymbol{G}_{i_k}\boldsymbol{Q}\right) \right) e^{\boldsymbol{Q}\left(t_{i_m}-t_{i_{m-1}}\right)}\boldsymbol{Q}\boldsymbol{G}_{i_m}\boldsymbol{1}
$$

where $t_{i_0} = 0$. This proves (3.1.13). □

3.2. **The marginal distributions.** In this section we state expressions for the marginal survival distributions $\mathbb{P}[\tau_i > t]$ and $\mathbb{P}[T_k > t]$, and for $\mathbb{P}[T_k > t, T_k = \tau_i]$ which is the probability that the k-th default is by obligor i and that it not occurs before t. The first ones are more or less standard, while the second one is less so. These marginal distributions are needed to compute single-name CDS spreads and k^{th}-to-default spreads, see e.g [17], [19]. Note that CDS-s are used as calibration instruments when pricing portfolio credit derivatives. We come back to this in Section 4. The following lemma is trivial, but stated since it is needed later on.

Lemma 3.5. *Consider m obligors with default intensities* (2.1). *Then,*

$$
\mathbb{P}[\tau_i > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{g}^{(i)} \quad \textit{and} \quad \mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\boldsymbol{m}^{(k)} \qquad (3.2.1)
$$

where the column vectors $\boldsymbol{g}^{(i)}$, $\boldsymbol{m}^{(k)}$ *of length* $|\boldsymbol{E}|$ *are defined as*

$$\boldsymbol{g}^{(i)}_{\boldsymbol{j}} = 1_{\left\{\boldsymbol{j}\in(\Delta_i)^C\right\}} \quad \text{and} \quad \boldsymbol{m}^{(k)}_{\boldsymbol{j}} = 1_{\left\{\boldsymbol{j}\in\cup_{n=0}^{k-1}\boldsymbol{E}_n\right\}}$$

and $\boldsymbol{E}_n$ *is set of states consisting of precisely* n *elements of* $\{1,\dots m\}$ *where* $\boldsymbol{E}_0 = \{0\}$.

The lemma immediately follows from the definition of τ_i in Proposition 2.1. The same holds for the distribution for T_k, where we also use that $\boldsymbol{m}^{(k)}$ sums the probabilities of states where there has been less than k defaults. We next restate the following result, proved in [19].

Proposition 3.6. *Consider* m *obligors with default intensities* (2.1). *Then,*

$$\mathbb{P}\left[T_k > t, T_k = \tau_i\right] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}\sum_{\ell=0}^{k-1}\left(\prod_{p=\ell}^{k-1}\boldsymbol{G}^{i,p}\boldsymbol{P}\right)\boldsymbol{h}^{i,k}, \tag{3.2.2}$$

for $k = 1,\dots m$, *where*

$$\boldsymbol{P}_{\boldsymbol{j},\boldsymbol{j}'} = \frac{\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}'}}{\sum_{\boldsymbol{k}\neq\boldsymbol{j}}\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{k}}}, \quad \boldsymbol{j},\boldsymbol{j}' \in \boldsymbol{E},$$

and $\boldsymbol{h}^{i,k}$ *is column vectors of length* $|\boldsymbol{E}|$ *and* $\boldsymbol{G}^{i,k}$ *is* $|\boldsymbol{E}|\times|\boldsymbol{E}|$ *diagonal matrices, defined by*

$$\boldsymbol{h}^{i,k}_{\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i\cap\boldsymbol{E}_k\}} \quad \text{and} \quad \boldsymbol{G}^{i,k}_{\boldsymbol{j},\boldsymbol{j}} = 1_{\left\{\boldsymbol{j}\in(\Delta_i)^C\cap\boldsymbol{E}_k\right\}} \quad \text{and} \quad \boldsymbol{G}^{i,k}_{\boldsymbol{j},\boldsymbol{j}'} = 0 \quad \text{if} \quad \boldsymbol{j}\neq\boldsymbol{j}'.$$

Equipped with the above distributions, we can derive closed-form solutions for single-name CDS spreads and k^{th}-to-default swaps for a nonhomogeneous portfolio, see e.g [17], [19]. In the present we focus on CDS spreads as our main calibration tools, see Section 4.

3.3. **The default correlations.** In this subsection we derive expressions for pairwise default correlations, i.e. $\rho_{i,j}(t) = \mathrm{Corr}(1_{\{\tau_i\le t\}}, 1_{\{\tau_j\le t\}})$ between the obligors $i\neq j$ belonging to a portfolio of m obligors satisfying (2.1).

Lemma 3.7. *Consider* m *obligors with default intensities* (2.1). *Then, for any pair of obligors* $i\neq j$,

$$\rho_{i,j}(t) = \frac{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{c}^{(i,j)} - \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(i)}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(j)}}{\sqrt{\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(i)}\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(j)}\left(1-\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(i)}\right)\left(1-\boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{h}^{(j)}\right)}} \tag{3.3.1}$$

where the column vectors $\boldsymbol{h}^{(i)}$, $\boldsymbol{c}^{(i,j)}$ *of length* $|\boldsymbol{E}|$ *are defined as*

$$\boldsymbol{h}^{(i)}_{\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i\}} \quad \text{and} \quad \boldsymbol{c}^{(i,j)}_{\boldsymbol{j}} = 1_{\{\boldsymbol{j}\in\Delta_i\cap\Delta_j\}} = \boldsymbol{h}^{(i)}_{\boldsymbol{j}}\boldsymbol{h}^{(j)}_{\boldsymbol{j}}. \tag{3.3.2}$$

Proof. By the definition of covariance and variance

$$\mathrm{Cov}(1_{\{\tau_i\le t\}}, 1_{\{\tau_j\le t\}}) = \mathbb{P}\left[\tau_i\le t, \tau_j\le t\right] - \mathbb{P}\left[\tau_i\le t\right]\mathbb{P}\left[\tau_j\le t\right],$$

and $\mathrm{Var}(1_{\{\tau_i \leq t\}}) = \mathbb{P}[\tau_i \leq t](1 - \mathbb{P}[\tau_i \leq t])$. According to Equation (2.2) we have that $\mathbb{P}[\tau_i \leq t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{h}^{(i)}$ where $\boldsymbol{h}_{\boldsymbol{j}}^{(i)} = 1_{\{\boldsymbol{j} \in \Delta_i\}}$, and that

$$\mathbb{P}[\tau_i \leq t, \tau_j \leq t] = \mathbb{P}[Y_t \in \Delta_i \cap \Delta_j] = \sum_{\boldsymbol{j} \in \Delta_i \cap \Delta_j} \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{c}^{(i,j)}$$

where $\boldsymbol{c}_{\boldsymbol{j}}^{(i,j)} = 1_{\{\boldsymbol{j} \in \Delta_i \cap \Delta_j\}} = \boldsymbol{h}_{\boldsymbol{j}}^{(i)} \boldsymbol{h}_{\boldsymbol{j}}^{(j)}$. Inserting these expressions into the definition for correlation between two random variables yields (3.3.1). □

Note that if we have determined the vector $\boldsymbol{g}^{(i)}$, then $\boldsymbol{h}^{(i)}$ is retrieved from $\boldsymbol{g}^{(i)}$ according to $\boldsymbol{h}^{(i)} = \boldsymbol{1} - \boldsymbol{g}^{(i)}$ which is useful for practical implementation.

3.4. **Expected default times.** By construction (see Proposition 2.1), the intensity matrix $\boldsymbol{Q}$ for the Markov jump process Y_t on $\boldsymbol{E}$ has the form

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0} & 0 \end{pmatrix}$$

where $\boldsymbol{t}$ is a column vector with $|\boldsymbol{E}| - 1$ rows. The $\boldsymbol{j}$-th element $\boldsymbol{t}_{\boldsymbol{j}}$ is the intensity for Y_t to jump from the state $\boldsymbol{j}$ to the absorbing state $\Delta = \cap_{i=1}^m \Delta_i$. Furthermore, $\boldsymbol{T}$ is invertible since it is upper diagonal with strictly negative diagonal elements. Thus, we have the following standard lemma.

Lemma 3.8. *Consider m obligors with default intensities* (2.1)*. Then, with notation as above*

$$\mathbb{E}[\tau_i^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-n} \tilde{\boldsymbol{g}}^{(i)} \quad \text{and} \quad \mathbb{E}[T_k^n] = (-1)^n n! \tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-n} \tilde{\boldsymbol{m}}^{(k)}$$

for $n \in \mathbb{N}$ where $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{g}}^{(i)}, \tilde{\boldsymbol{m}}^{(k)}$ are the restrictions of $\boldsymbol{\alpha}, \boldsymbol{g}^{(i)}, \boldsymbol{m}^{(k)}$ from $\boldsymbol{E}$ to $\boldsymbol{E} \setminus \Delta$.

Proof. We prove the results for $n = 1$. By Lemma 3.5 we have that

$$\mathbb{P}[\tau_i > t] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T}t} \tilde{\boldsymbol{g}}^{(i)} \quad \text{and} \quad \mathbb{P}[T_k > t] = \tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T}t} \tilde{\boldsymbol{m}}^{(k)}$$

where $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{g}}^{(i)}, \tilde{\boldsymbol{m}}^{(k)}$ are the restrictions of $\boldsymbol{\alpha}, \boldsymbol{g}^{(i)}, \boldsymbol{m}^{(k)}$ from $\boldsymbol{E}$ to $\boldsymbol{E} \setminus \Delta$. If $F_{T_k}(t) = \mathbb{P}[T_k \leq t]$, then $f_{T_k}(t)$ is given by

$$f_{T_k}(t) = \frac{d}{dt} F_{T_k}(t) = -\frac{d}{dt} \mathbb{P}[T_k > t] = -\tilde{\boldsymbol{\alpha}} e^{\boldsymbol{T}t} \boldsymbol{T} \tilde{\boldsymbol{m}}^{(k)}$$

so that

$$\mathbb{E}[T_k] = \int_0^\infty t f_{T_k}(t)\, dt = -\tilde{\boldsymbol{\alpha}} \int_0^\infty t e^{\boldsymbol{T}t} dt \boldsymbol{T} \tilde{\boldsymbol{m}}^{(k)} = -\tilde{\boldsymbol{\alpha}} \boldsymbol{T}^{-1} \tilde{\boldsymbol{m}}^{(k)}.$$

To motivate the last equality we use partial integration and the fact that $\boldsymbol{T}$ is invertible to conclude that

$$\int_0^\infty t e^{\boldsymbol{T}t} dt \boldsymbol{T} = \lim_{t \to \infty} e^{\boldsymbol{T}t} \left(t\boldsymbol{I} - \boldsymbol{T}^{-1} \right) + \boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}$$

since $\lim_{t \to \infty} e^{\boldsymbol{T}t} \left(t\boldsymbol{I} - \boldsymbol{T}^{-1} \right) = \boldsymbol{0}$ because the eigenvalues of $\boldsymbol{T}$ are strictly negative. The expression for $\mathbb{E}[T_k^n]$ and $\mathbb{E}[\tau_i^n]$ are derived analogously for $n = 1, 2, 3, \ldots$. □

The above proof can also be done by using Laplace transforms, see e.g. [2]. From Lemma 3.8 we can determine the risk-neutral, i.e implied, expected default times according to $\mathbb{E}[\tau_i] = -\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{g}}^{(i)}$ and $\mathbb{E}[T_k] = -\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{m}}^{(k)}$. Furthermore, the implied variances of the default times are then given by

$$\mathrm{Var}[\tau_i] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{g}}^{(i)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{g}}^{(i)}\right)^2 \qquad \text{for } i = 1, 2, \ldots, m$$
$$\mathrm{Var}[T_k] = 2\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-2}\tilde{\boldsymbol{m}}^{(k)} - \left(\tilde{\boldsymbol{\alpha}}\boldsymbol{T}^{-1}\tilde{\boldsymbol{m}}^{(k)}\right)^2 \qquad \text{for } k = 1, 2, \ldots, m.$$

3.5. **Some remarks.** The message in Subsections 3.2-3.3 is that under (2.1), computations of multivariate default and survival distributions, conditional multivariate default and survival distributions, marginal default distributions, multivariate default densities and default correlations can be reduced to compute the matrix exponential. Computing $e^{\boldsymbol{Q}t}$ efficiently is a numerical issue, which for large state spaces requires special treatment. This is discussed in Section 6. Finally, recall that $|\boldsymbol{E}| = 2^m$ which in practice will force us to work with portfolios where m is less or equal to 25, say ([19] used $m = 15$).

4. Calibrating the model parameters against CDS spreads and CDS correlations

In this section we discuss how to find the parameters in the model (2.1). First, Subsection 4.1 derives the model spreads for single-name credit default swaps, CDS-s, which are the most liquid traded credit derivative today. Next, Subsection 4.2 gives a detailed description of the calibration against CDS spreads and the corresponding CDS-correlations. We also discuss how to deal with negative jumps in the intensities, which are required if there are negative CDS-correlations

4.1. **Using the matrix-analytic approach to find CDS spreads.** Given the model (2.1), we will in this subsection derive expressions for CDS-spreads, which constitute our primary calibration instruments. In the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities.

Consider a single-name credit default swap (CDS) with maturity T where the reference entity is a obligor i with default times τ_i and recovery rates ϕ_i. The protection premiums are paid at $0 < t_1 < t_2 < \ldots < t_{n_T} = T$ if $\tau_i > T$, or until the default time of obligor i, whichever comes first. Assuming that the default time and the risk-free interest rate are independent for each obligor and that the recovery rate is deterministic, one can show that the CDS spread is given by (see e.g. [17] or [19]),

$$R_i(T) = \frac{(1-\phi_i)\int_0^T B_s dF_i(s)}{\sum_{n=1}^{n_T}\left(B_{t_n}\Delta_n(1-F_i(t_n)) + \int_{t_{n-1}}^{t_n} B_s\,(s-t_{n-1})\,dF_i(s)\right)} \tag{4.1.1}$$

where $B_t = \exp\left(-\int_0^t r_s ds\right)$ denote the discount factor, r_t is the risk-free interest rate, and $F_i(t) = \mathbb{P}[\tau_i \leq t]$ is the distribution function of the default time for obligor i. Note that the CDS spread is independent of the amount that is protected. Expressions for $R_i(T)$ may be obtained by inserting the expression for $\mathbb{P}[\tau_i > t]$ in (3.2.1) into (4.1.1), and have previously been stated in [18], [19], but without proofs. For completeness, this is done in the following proposition.

Proposition 4.1. *Consider m obligors with default intensities* (2.1) *and assume that the interest rate r is constant. Then,*

$$R_i(T) = \frac{(1-\phi_i)\boldsymbol{\alpha}\left(\boldsymbol{A}(0)-\boldsymbol{A}(T)\right)\boldsymbol{g}^{(i)}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n}+\boldsymbol{C}(t_{n-1},t_n)\right)\right)\boldsymbol{g}^{(i)}} \tag{4.1.2}$$

where $\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t)-\boldsymbol{A}(s)\right)-\boldsymbol{B}(t)+\boldsymbol{B}(s)$ *for* $\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$ *and*

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I}+\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Proof. Let $f_i(t)$ denote the density for τ_i,

$$f_i(t) = \frac{d}{dt}F_i(t) = -\frac{d}{dt}\mathbb{P}[\tau_i > t] = -\boldsymbol{\alpha}\boldsymbol{Q}e^{\boldsymbol{Q}t}\boldsymbol{g}^{(i)}$$

where the last equality is due to Lemma 3.5. Then,

$$\int_0^T B_t dF_i(t) = \int_0^T e^{-rt}f_i(t)dt = -\boldsymbol{\alpha}\int_0^T \boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\boldsymbol{g}^{(i)} = \boldsymbol{\alpha}\left(\boldsymbol{A}(0)-\boldsymbol{A}(T)\right)\boldsymbol{g}^{(i)}$$

since

$$\int_a^b \boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \boldsymbol{A}(b)-\boldsymbol{A}(a) \quad \text{where} \quad \boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Furthermore,

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} B_t\left(t-t_{n-1}\right)dF_i(t) &= \int_{t_{n-1}}^{t_n} te^{-rt}f_i(t)dt - t_{n-1}\int_{t_{n-1}}^{t_n} e^{-rt}f_i(t)dt \\
&= -\boldsymbol{\alpha}\left(\int_{t_{n-1}}^{t_n} t\boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt - t_{n-1}\int_{t_{n-1}}^{t_n}\boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt\right)\boldsymbol{g}^{(i)} \\
&= \boldsymbol{\alpha}\left(t_{n-1}\left(\boldsymbol{A}(t_n)-\boldsymbol{A}(t_{n-1})\right)-\boldsymbol{B}(t_n)+\boldsymbol{B}(t_{n-1})\right)\boldsymbol{g}^{(i)} \\
&= \boldsymbol{\alpha}\boldsymbol{C}(t_{n-1},t_n)\boldsymbol{g}^{(i)}
\end{aligned}$$

where $\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t)-\boldsymbol{A}(s)\right)-\boldsymbol{B}(t)+\boldsymbol{B}(s)$ and

$$\int_a^b t\boldsymbol{Q}e^{(\boldsymbol{Q}-r\boldsymbol{I})t}dt = \boldsymbol{B}(b)-\boldsymbol{B}(a) \quad \text{for} \quad \boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I}+\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q}-r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}.$$

Now, inserting the above expressions in Equation (4.1.1) renders (4.1.2). □

By using the technique in Proposition 4.1 and the expressions for $\mathbb{P}[T_k > t, T_k = \tau_i]$ and $\mathbb{P}[T_k > t]$ in Subsection 3.2, we can derive formulas for k^{th}-to-default swaps, which are generalizations of CDS contracts, to a portfolio of several obligors. These contracts offers protection on the k^{th} default in the portfolio. For more on this, see e.g. [17], [19].

4.2. **The calibration.** The parameters in (2.1) are obtained by calibrating the model against market CDS spreads and market CDS correlations. As in [19] we reparameterize the basic description (2.1) of the default intensities to the form

$$\lambda_{t,i} = a_i \left(1 + \sum_{j=1, j\neq i}^{m} \theta_{i,j} 1_{\{\tau_j \leq t\}} \right), \tag{4.2.1}$$

where the a_i-s are the base default intensities and the $\theta_{i,j}$ measure the "relative dependence structure". In [19] we assumed that the matrix $\{\theta_{i,j}\}$ was exogenously given and then calibrated the a_i-s against the m market CDS spreads. In this paper we use the m market CDS spreads as in [19] but in addition also determine the $\{\theta_{i,j}\}$ from market data. Let $\rho_{i,j}(T) = \text{Corr}(1_{\{\tau_i \leq T\}}, 1_{\{\tau_j \leq T\}})$ be the default correlation matrix computed under the risk neutral measure. This matrix is a function of the parameters $\{\theta_{i,j}\}$, but is not observable. Instead we use $\beta\{\rho_{i,j}^{(\text{CDS})}(T)\}$ as a proxy for it, where $\{\rho_{i,j}^{(\text{CDS})}(T)\}$ is the observed correlation matrix for the T-years market CDS spreads, and β is a parameter at our disposal. Thus, in the calibration we match $\rho_{i,j}(T)$ against $\beta\{\rho_{i,j}^{(\text{CDS})}(T)\}$.

For standardized portfolios, CDS-correlation matrices can be obtained from e.g. Reuters. However, given times-series for the CDS-spreads on obligors in any portfolio, these matrices can easily be computed using standard mathematical software.

A further issue remains. This is that the CDS correlation matrix is symmetric and thus only contains $m(m-1)/2$ pairwise CDS correlations. Hence, together with the m market CDS spreads we have $m(m+1)/2$ data observations, while there are m^2 unknown parameters in (4.2.1); the $m(m-1)$ different $\theta_{i,j}$-s and the m base intensities $\{a_i\}$. To make the number of model parameters and the number of market observations match, we hence assume that the $\theta_{i,j}$-s are the same for some of the ordered pairs (i,j), so that there are only $m(m-1)/2$ different $\theta_{i,j}$-s.

We now explain the calibration in more detail. First, we reduce the $m(m-1)$ unknown variables $\{\theta_{i,j}\}$ to a set of $(m-1)m/2$ different nonnegative parameters $\{d_q\} = \{d_1, d_2, \ldots d_{(m-1)m/2}\}$, so that the total number of model parameters are as many as the market observations. Secondly, we assume a exogenously given dependence matrix $\{D_{i,j}\}$ where $D_{i,j} \in \{1, 2, \ldots (m-1)m/2\}$ which determines the matrix $\{\theta_{i,j}\}$ according to $\theta_{i,j} = \pm d_{D_{i,j}}$, where the sign is the same as the market CDS correlation $\rho_{i,j}^{(\text{CDS})}(T)$. It is a topic for future research to find methods to estimate the dependence matrix $\{D_{i,j}\}$. For example, from corporate data or from the rapidly increasing market of credit portfolio products, such as CDO's and basket default swaps. In this paper, the matrix $\{D_{i,j}\}$ is determined randomly, see Appendix 8.

Let $\boldsymbol{v} = (\{a_i\}, \{d_q\})$ denote the parameters describing the model and let $\{R_i(T; \boldsymbol{v})\}$ be the m different model T-year CDS spreads and $\{R_{i,M}(T)\}$ the corresponding market spreads. Furthermore, as above, we let $\rho_{i,j}(T; \boldsymbol{v}) = \text{Corr}(1_{\{\tau_i \leq T\}}, 1_{\{\tau_j \leq T\}})$ denote the pairwise T-year default correlations. Here we have emphasized that the model quantities are functions of $\boldsymbol{v} = (\{a_i\}, \{d_q\})$ but suppressed the dependence of the matrix $\{D_{i,j}\}$, interest rate, payment frequency, etc. The vector $\boldsymbol{v}$ is obtained as

$$\boldsymbol{v} = \underset{\hat{\boldsymbol{v}}}{\operatorname{argmin}} \left[\delta_{\text{CDS}}(T;\hat{\boldsymbol{v}}) + \delta_{\text{corr}}(T;\hat{\boldsymbol{v}})\right] \tag{4.2.2}$$

where

$$\begin{aligned}\delta_{\text{CDS}}(T;\boldsymbol{v}) &= F\sum_{i=1}^{m}\left(R_i(T;\boldsymbol{v}) - R_{i,M}(T)\right)^2 \\ \delta_{\text{corr}}(T;\boldsymbol{v}) &= \sum_{i=1}^{m}\sum_{j=i+1}^{m}\left(\rho_{i,j}(T;\boldsymbol{v}) - \beta\rho_{i,j}^{(\text{CDS})}(T)\right)^2\end{aligned} \tag{4.2.3}$$

with $F > 0$ and $0 < \beta \leq 1$ exogenously chosen. The second expression in (4.2.3) is due to that we use $\beta\{\rho_{i,j}^{(\text{CDS})}(T)\}$ as a proxy for $\{\rho_{i,j}(T)\}$. It is possible to include F and β in the unknown parameter vector $\boldsymbol{v}$ and we make some further comments on this at the end of the present subsection.

If all CDS-correlations are positive, the minimization in (4.2.2) is performed with the constraint that all elements in $\boldsymbol{v}$ are nonnegative. However, if there are negative CDS-correlations, that is $\rho_{i,j}^{(\text{CDS})}(T) < 0$ for some pairs (i,j), then we require that $\theta_{i,j} = \text{sign}(\rho_{i,j}^{(\text{CDS})}(T))d_{D_{i,j}} = -d_{D_{i,j}} < 0$, since it otherwise is difficult to generate negative default correlations. Because $\lambda_{t,i}$ must be positive and all parameters are nonnegative, we have to bound some of the $\{d_q\}$ if there are negative CDS-correlations. It is then practical to assume that the dependence matrix $\{D_{i,j}\}$ is constructed so that it splits $\{d_q\}$ in two disjoint groups, $\{d_q\} = \boldsymbol{d}_- \cup \boldsymbol{d}_+$ such that if $\rho_{i,j}^{(\text{CDS})}(T) < 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and if $\rho_{i,j}^{(\text{CDS})}(T) \geq 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_+$. Let N_i denote the sets of obligors $j \neq i$ which are negatively correlated with entity i, that is, where $\rho_{i,j}^{(\text{CDS})}(T) < 0$. Thus, if $j \in N_i$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and the following constraints

$$a_i - \sum_{j\in N_i} a_i d_{D_{i,j}} > 0 \quad \text{that is,} \quad 1 > \sum_{j\in N_i} d_{D_{i,j}}, \tag{4.2.4}$$

must simultaneously hold for every $i = 1, 2, \ldots, m$. These joint bounds finally determine the proper constraints on the parameters in $\boldsymbol{d}_-$, which heavily depend on the elements $D_{i,j}$ and the sign of $\rho_{i,j}^{(\text{CDS})}(T)$. If the number of negative CDS correlations are less than positive CDS correlations, it may be convenient to assume that each p, where $d_p \in \boldsymbol{d}_-$, only appears once in the matrix $\{D_{i,j}\}$ and use the constraints $d_p < \frac{1}{|N_i|}$ if $\theta_{i,j} = -d_p$ for some $j \in N_i$. Recall that in economic terms, negative CDS correlation, and thus negative jumps in the intensities for a obligor i, means that entity i benefits from defaults of obligors $j \in N_i$.

Let us finally give some remarks on the parameters β and F. A naive first attempt is to let $F = 1$ and $\beta = 1$ in the calibration (4.2.2). However, the market CDS spreads $R_{i,M}(T)$ are about 100 times smaller than $\rho_{i,j}^{(\text{CDS})}(T)$, which then implies unrealistic model CDS spreads. The problem can be avoided by letting $\sqrt{F} = 100$ so that $\sqrt{F}R_{i,M}(T)$ and $\rho_{i,j}^{(\text{CDS})}(T)$ are approximately in the same order. This leads to bad correlation fits, i.e. $\delta_{\text{corr}}(T;\boldsymbol{v})$ is big, when $\beta = 1$. In our examples, the calibrations produce default correlations much smaller than the corresponding CDS correlations. Motivated by this we assume that $0 < \beta << 1$ and in our numerical studies we let $\beta = 0.05$ and $\sqrt{F} = 100$. This gives perfect correlation calibrations for our data sets where all entities in the CDS-correlation matrix are nonnegative, and reasonable calibrations when the correlation matrix contains both positive and negative entities (see Subsection 5.1). It is possible to include β and F in the parameter vector $\boldsymbol{v}$, and then decrease the set $\{d_q\}$ so that $|\{d_q\}| = m(m-1)/2-2$, where the total number of model parameters still are as many as the market observations.

5. Numerical studies

In this section we will use the theory developed in previous sections to study quantities of importance in active credit portfolio management. We consider the same parameterization of (2.1) as in Subsection 4.2, that is

$$\lambda_{t,i} = a_i \left(1 + \sum_{j=1, j\neq i}^{m} \varepsilon_{i,j} d_{D_{i,j}} 1_{\{\tau_j \leq t\}} \right),$$

where $\varepsilon_{i,j}$ is the sign of $\rho_{i,j}^{(\mathrm{CDS})}(T)$, and $\{D_{i,j}\}$ is a exogenously given matrix such that $D_{i,j} \in \left\{1, 2 \ldots \frac{(m-1)m}{2}\right\}$. Further, the d_q-s are $(m-1)m/2$ different nonnegative parameters which will be determined in the calibration, together with the base default intensities a_i.

In Subsection 5.1 we introduce two CDS portfolio, one in the European auto sector, the other in the European financial sector. These portfolios, which both consist of 10 companies, are used as a basis for the numerical studies in the rest of this section. For exogenously given dependence matrices $\{D_{i,j}\}$, we calibrate the portfolios against market CDS spreads and their correlations. In the calibrated portfolios, we then study the implied joint default and survival distributions and the implied univariate and bivariate conditional survival distributions (Subsection 5.2), the implied default correlations (Subsection 5.3), and finally the implied expected default times and expected ordered defaults times (Subsection 5.4).

5.1. **Two CDS portfolios.** Table 1 and Table 2 describe the two CDS portfolios which are used in our numerical studies and Table 3 and Table 4 their correlation matrices. The maturity was 5 years and the data was obtained from `Reuters` at February 15, 2007 for the auto portfolio and March 28, 2007 for the financial portfolio.

The correlation matrices are based on rolling 12 months 5-years CDS midpoint market spreads for each obligor, with a daily sampling frequency of the closing level of the spreads. In both portfolios, we have assumed a fictive relative dependence structure $\{D_{i,j}\}$ which are given in Table 11 and Table 12 in Appendix 8 together with a description how they where created. Further, we have also assumed a fictive recovery rate structure which is the same in both baskets. The interest rate was assumed to be constant and set to 3%, and the protection fees were assumed to be paid quarterly.

For each portfolio, the a_i-s and d_q-s are obtained by simultaneously calibrate the CDS spreads in Table 1 and Table 2 and the corresponding correlation matrices in Table 3 and Table 4, as described in Subsection 4.2. In both portfolios the CDS calibrations

Table 1: The auto companies and their 5 year market (2007-02-15) and model CDS spreads, the absolute calibration errors, and the recoveries. The spreads are given in bp.

Company name	Market	Model	abs.error	recovery %
Volvo AB	25.84	25.87	0.03336	32
BMW AG	9.415	9.593	0.178	48
Comp. Fi. Michelin SA	25.34	25.53	0.1915	45
Continental AG	43.66	43.68	0.01789	34
DaimlerChrysler AG	44	43.98	0.02175	42
Fiat SPA	58	58.02	0.016	41
Peugeot SA	24.84	24.9	0.06289	29
Renault SA	28.67	28.72	0.05989	39
Valeo SA	66	65.98	0.01812	51
Volkswagen AG	22.17	22.08	0.08343	41
Σ abs.cal.err			0.6828 bp	

Table 2: The financial companies and their 5 year market (2007-03-28) and model CDS spreads, the absolute calibration errors, and the recoveries. The spreads are given in bp.

Company name	Market	Model	abs.error	recovery %
ABN Amro Bank NV	6.085	6.225	0.1402	32
Barclays Bank PLC	7	6.9	0.1	48
BNP Paribas	6.665	6.562	0.1026	45
Commerzbank AG	9.335	9.41	0.07492	34
Deutsche Bank AG	13.59	13.5	0.08747	42
HSBC Bank PLC	7.25	7.247	0.002626	41
Hypovereinsbank AG	7	7.217	0.2173	29
The Royal Bank of Scotland PLC	7	6.844	0.1556	39
Banco Santander Central Hispano	8.25	8.22	0.02998	51
Unicredito Italiano SPA	9.915	9.989	0.07363	41
Σ abs.cal.err			0.9844 bp	

where perfect. The correlation fit for the financial portfolio was also perfect, as seen in Table 5, while the corresponding calibration for the auto case was mediocre. One possible explanation for the lesser performance in the auto portfolio, is that the negative jumps in the intensities are bounded, which may bound the absolute value of the negative CDS-correlations by a scalar smaller than one.

A quick look in Table 14 reveals that 15 (out of 18) "negative" parameters hit their upper bounds (for more details on this, see Appendix). Such limitations can be avoided by using a different parametrization of the intensities in (2.1), making the jumps-sizes also

Table 3: The auto CDS correlation matrix, based on 5-years CDS midpoint market spreads for each obligor, between 2006-02-15 and 2007-02-15, with a daily sampling frequency of the closing level of the spreads.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	1									
BMW	0.63	1								
MICH	0.81	0.64	1							
CONT	-0.5	-0.69	-0.23	1						
DCX	0.12	0.47	0.51	0.13	1					
FIAT	0.67	0.97	0.76	-0.64	0.52	1				
PEUG	0.66	0.28	0.81	0.14	0.34	0.37	1			
RENA	0.55	0.24	0.79	0.1	0.42	0.39	0.82	1		
VALE	0.22	-0.42	0.2	0.44	-0.1	-0.31	0.39	0.41	1	
VW	0.12	0.66	0.47	-0.2	0.77	0.71	0.16	0.34	-0.44	1

Table 4: The financial CDS correlation matrix, based on 5-years CDS midpoint market spreads for each obligor, between 2006-03-28 and 2007-03-28, with a daily sampling frequency of the closing level of the spreads.

	ABN	BACR	BNP	CMZB	DB	HSBC	HVB	RBOS	BSCH	CRDIT
ABN	1									
BACR	0.91	1								
BNP	0.98	0.94	1							
CMZB	0.92	0.95	0.92	1						
DB	0.88	0.84	0.89	0.81	1					
HSBC	0.66	0.96	0.76	0.9	0.88	1				
HVB	0.82	0.9	0.89	0.89	0.8	0.85	1			
RBOS	0.93	0.98	0.95	0.94	0.85	0.98	0.88	1		
BSCH	0.84	0.95	0.89	0.95	0.78	0.88	0.89	0.92	1	
CRDIT	0.78	0.9	0.82	0.91	0.76	0.81	0.87	0.84	0.96	1

be functions of the level of the intensity. To be more specific, the bigger the intensity, the bigger negative jumps are allowed.

From Table 14 and Table 15 in Appendix, we see that in the auto portfolio, the base intensities can have positive jumps up to 589% of their "base values" a_i, and up to 1749 % in the financial portfolio.

5.2. **The implied default and survival distributions and the conditional survival distributions.** In the credit literature today, risk-neutral distributions are often called *implied* distributions. Here "implied" is refereing to the fact that the quantities are retrieved

Table 5: The average, median, min and max absolute calibration-errors in percent of the scaled market CDS-correlations, i.e. $\{\beta\rho_{i,j}^{(\text{CDS})}(T)\}$, where $\beta = 0.05$

Portfolio	mean	median	min	max
Auto	29.2	18.8	0.347	122
Financial	1.43	0.213	0.00988	13.1

from market data via a model. The implied (joint) default and survival distributions at different time points, are important quantities for a credit manager. The results in Section 3 provides formulas for computing these expressions. In this subsection we use them to find the implied default and survival distributions, as well as conditional survival distributions, for different pairs of obligors, in the calibrated portfolios.

We want to study the bivariate default and survival distributions for the pairs Fiat, BMW and Continental, BMW. Given the CDS spreads and their correlations, it may in general be difficult to draw some qualitative conclusions about these bivariate probabilities and their mutual relations, without actually computing them. The CDS spreads for Fiat and BMW are positively correlated while Continental and BMW are negatively correlated, and the difference in percent between the spreads for Continental and Fiat are $(58 - 43.66)/58 = 24\%$. From this, we intuitively guess that BMW-s bivariate default probabilities with Fiat should be bigger than the bivariate default probabilities with Continental. Conversely, the bivariate survival distributions of the pair Fiat, BMW should be smaller than for Continental, BMW. These hypothesis are confirmed by the Figures 1, 2, 3 and 4. Similar shapes of the bivariate default and survival distributions are obtained by obligors in the financial portfolio, as seen in Figure 5 and 6.

We also note that the CDS spreads for Continental is positively correlated with the spreads for DaimlerChrysler, Peugeot, Renault and Valeo. We therefore suspect that the conditional survival distributions for continental are decreasing with the number of defaults among DaimlerChrysler, Peugeot, Renault and Valeo. For example, when s is fixed, we guess that the survival distribution $\mathbb{P}\left[\tau_{\text{Cont}} > t \,|\, \tau_{\text{DCX}} < s\right]$ as function of t for $t > s$, should lie above the curve $\mathbb{P}\left[\tau_{\text{Cont}} > t \,|\, \tau_{\text{DCX}} < s, \tau_{\text{Peu}} < s\right]$. This claim is supported by Figure 7 for $s = 10$ and $10 \leq t \leq 104$ (and also by Figure 9, for a similar test in the financial portfolio). Furthermore, the CDS spreads for Continental are negatively correlated with the spreads for Volvo, BMW, Michelin, Fiat and Volkswagen. In view of the above results, it is tempting to believe that the conditional survival distributions for continental, are increasing with the number of defaults among for Volvo, BMW, Michelin, Fiat and Volkswagen.

We investigate this for $s = 10$ and $10 \leq t \leq 104$, and note that the claim is only true on the interval $10 \leq t \leq 45$, as seen in Figure 8. For $t > 53$, we see that the curves do not lie in increasing order with increasing amount of negatively correlated defaults. One possible explanation for this is that the negative jumps in the intensities where bounded, in the specification that we use, which implies that the effect of a negative jump will diminish as time progress since several positive jumps then have occurred previously.

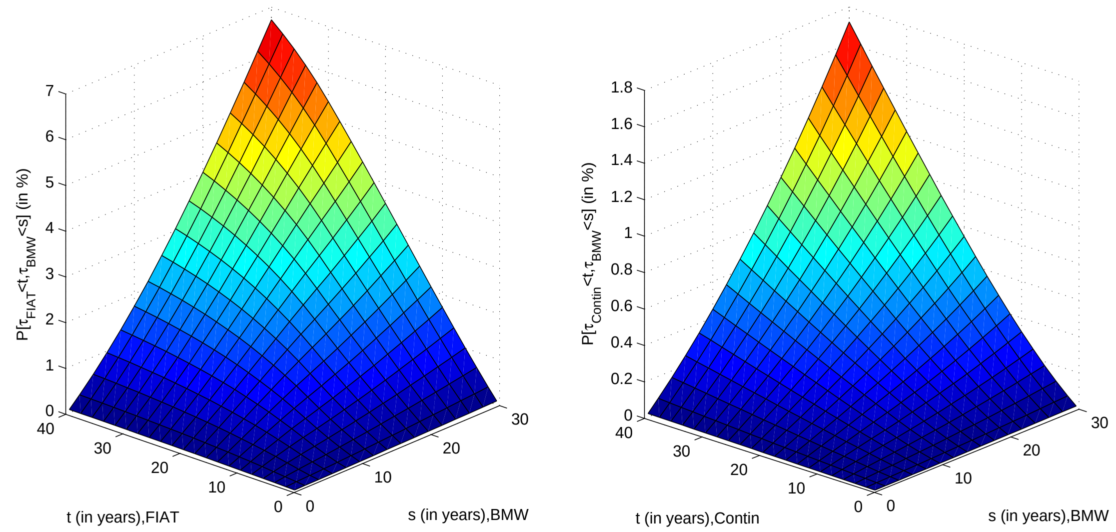

Figure 1: The implied bivariate default distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

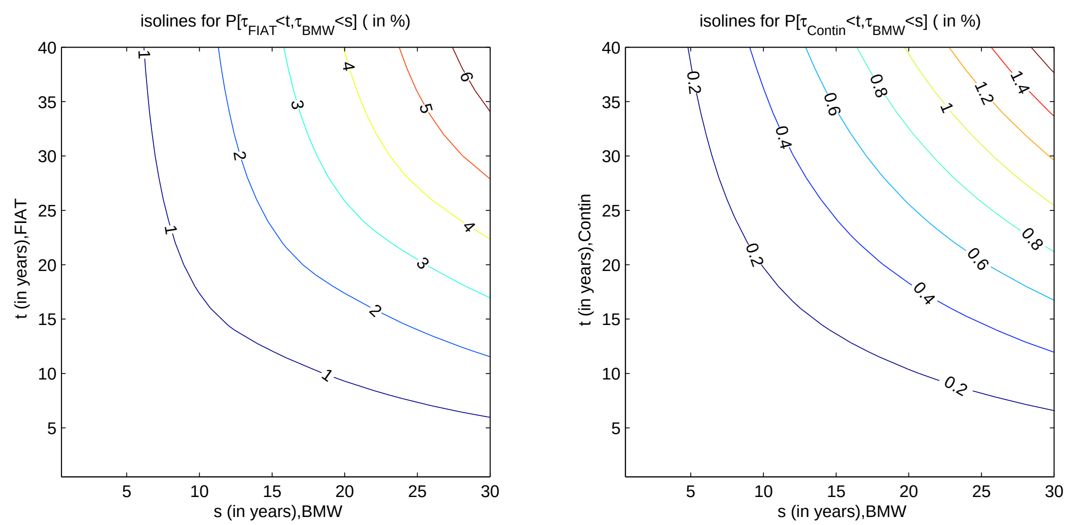

Figure 2: The isolines for the implied bivariate default distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

We also compare univariate conditional survival distribution, with bivariate conditional survival distribution, in the banking portfolio. In Figure 9 and Figure 10 we see that

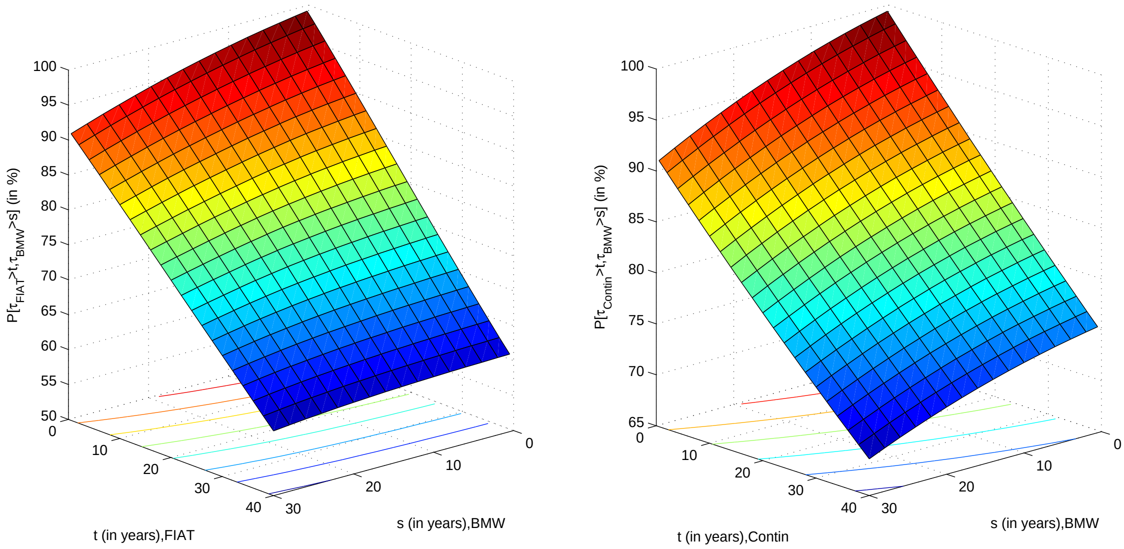

Figure 3: The implied bivariate survival distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

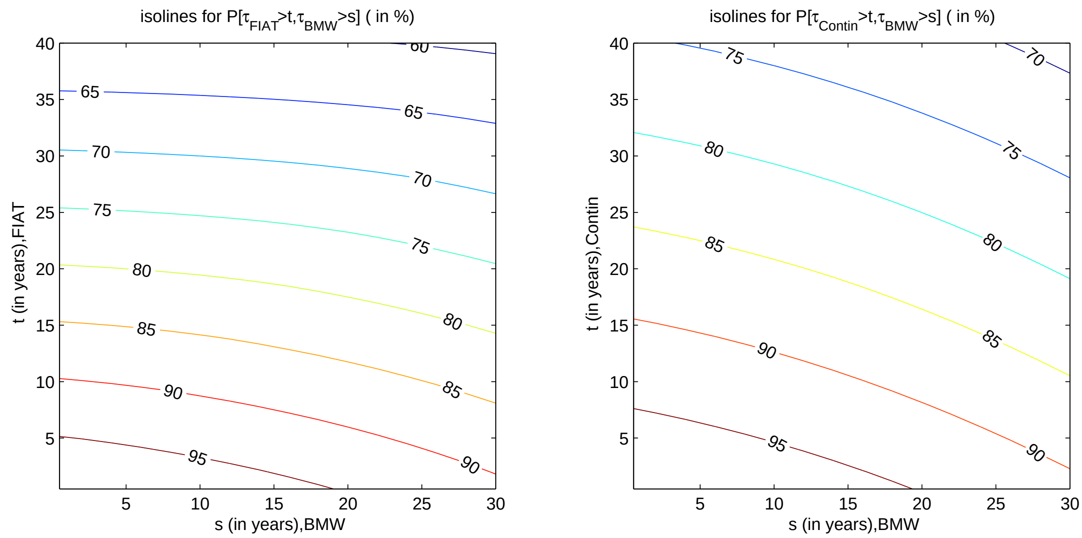

Figure 4: The isolines for the implied bivariate survival distribution for Fiat and BMW (left) and Continental and BMW (right) in the auto portfolio.

the bivariate conditional survival distribution declines much faster than the corresponding univariate conditional survival distribution.

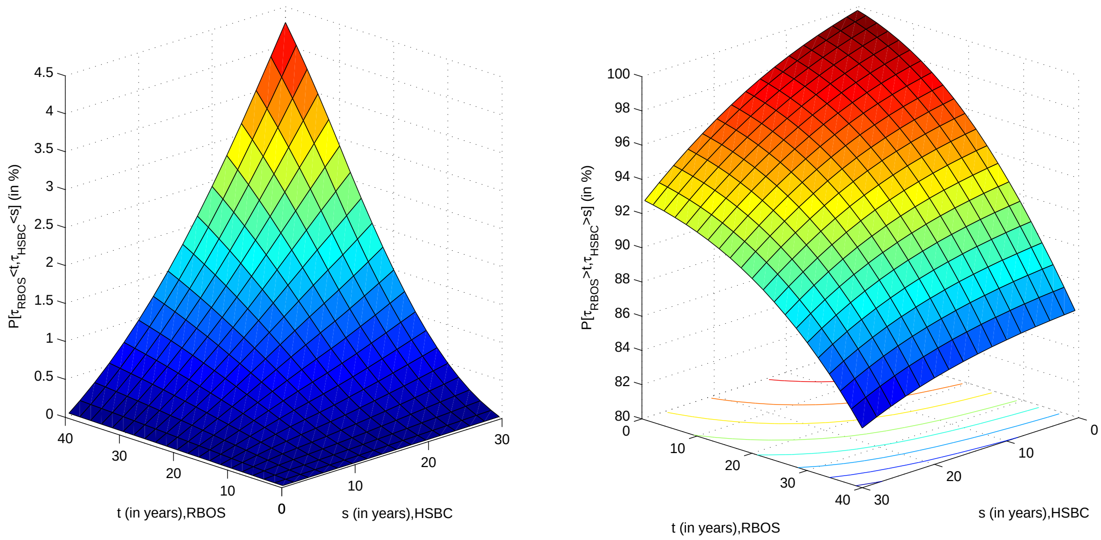

Figure 5: The implied bivariate default (left) and survival (right) distributions for Royal Bank of Scotland and HSBC Bank in the financial portfolio.

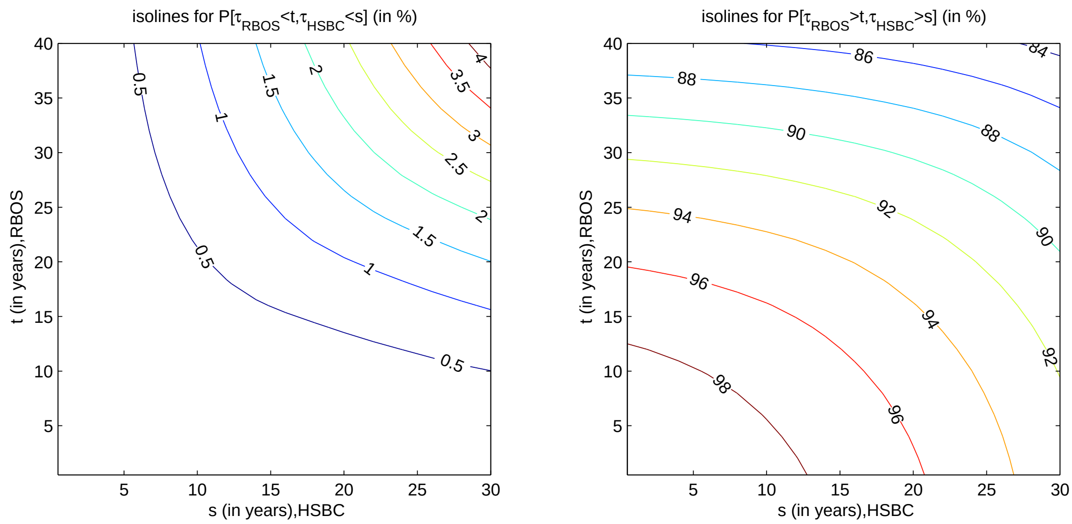

Figure 6: The isolines for the implied bivariate default (left) and survival (right) distributions for Royal Bank of Scotland and HSBC Bank in the financial portfolio.

So far we have only computed joint bivariate distributions, or distributions involving two time points. To show that we can handle distributions with all 10 obligors for 10

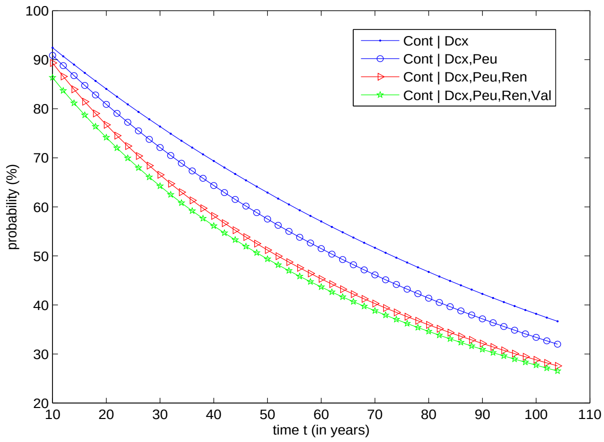

Figure 7: The survival distribution for Continental, conditional on defaults before time 10 years, by firms which are positively correlated with Continental. The firms which have defaulted are indicated in the legend.

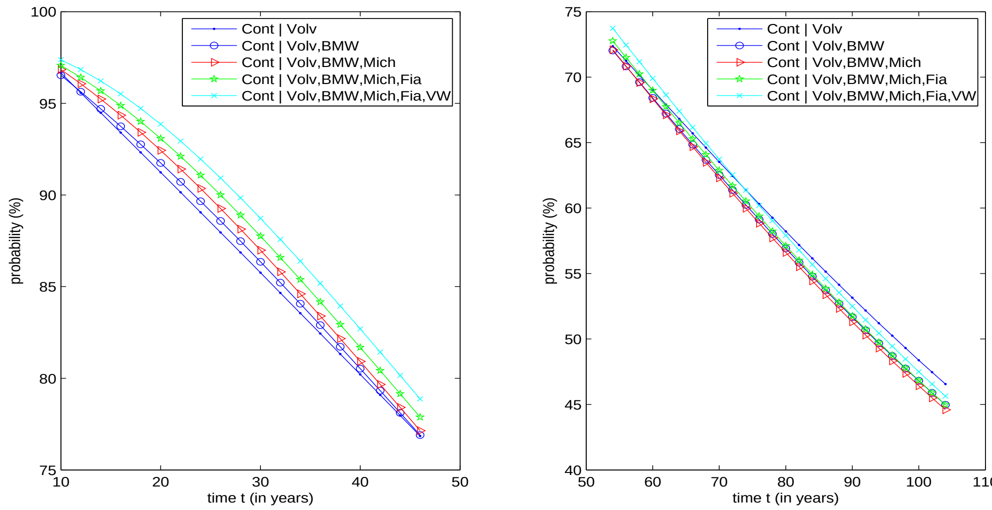

Figure 8: The survival distribution for Continental, conditional on defaults before time 10 years, by firms which are negatively correlated with Continental. Left figure $t < 45$, right figure $t > 52$. The firms which have defaulted are indicated in the legend.

different time points, Table 6 and 7 displays the joint multivariate default and survival distributions for all obligors, in each portfolio. Recall that implied default probabilities are often substantially larger then the "real" so called actuarial default probabilities.

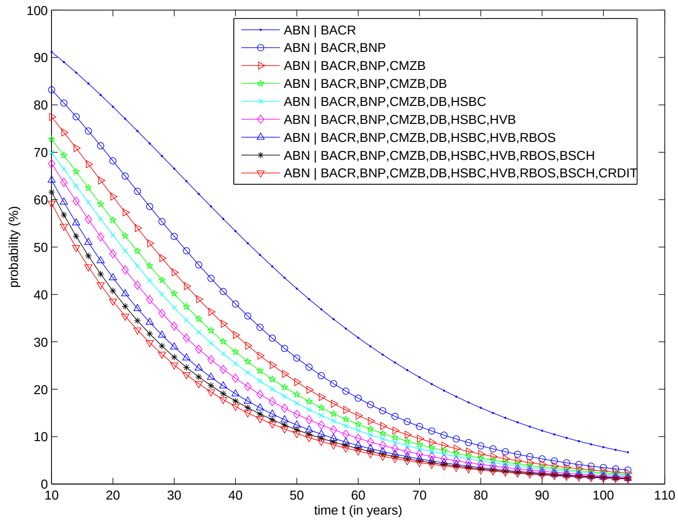

Figure 9: The survival distribution for ABN Amro, conditional on defaults before time 10 years. The firms which have defaulted are indicated in the legend.

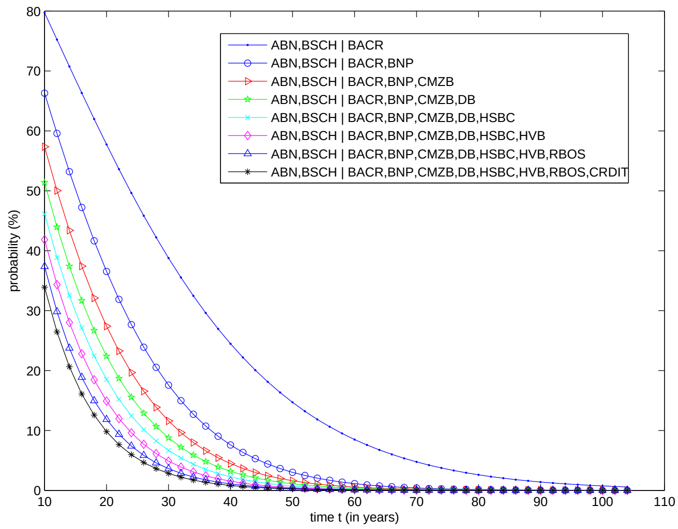

Figure 10: The joint survival distribution for ABN Amro and BSCH, conditional on defaults before time 10 years. The firms which have defaulted are indicated in the legend.

Table 6: The multivariate default and survival probabilities $\mathbb{P}[\tau_{\text{Volv}} > n, \dots, \tau_{\text{VW}} > 10n]$ and $\mathbb{P}[\tau_{\text{Volv}} \le n, \tau_{\text{BMW}} \le 2n, \dots, \tau_{\text{VW}} \le 10n]$ (in %) where $n = 0.5, 1, 1.5, \dots, 4$, in the auto portfolio.

	$n = 0.5$	$n = 1$	$n = 1.5$	$n = 2$	$n = 2.5$	$n = 3$	$n = 3.5$	$n = 4$
multi.def.prob	11.1	21	29.8	37.6	44.5	50.7	56.2	61
multi.surv.prob	98.3	96.6	94.9	93.1	91.4	89.7	88	86.2

Table 7: The multivariate default and survival probabilities $\mathbb{P}[\tau_{\text{ABN}} > n, \dots, \tau_{\text{CDRIT}} > 10n]$ and $\mathbb{P}[\tau_{\text{ABN}} \le n, \tau_{\text{BACR}} \le 2n, \dots, \tau_{\text{CDRIT}} \le 10n]$ (in %) where $n = 0.5, 1, 1.5, \dots, 4$, in the financial portfolio.

	$n = 0.5$	$n = 1$	$n = 1.5$	$n = 2$	$n = 2.5$	$n = 3$	$n = 3.5$	$n = 4$
multi.def.prob	29.4	49.3	63.3	73.3	80.4	85.6	89.5	92.3
multi.surv.prob	99.3	98.6	97.9	97.3	96.7	96.1	95.5	95

5.3. **The implied default correlations.** It may be of interest for a credit manager to have a quantitative grasp of the pairwise default correlations $\rho_{i,j}(t) = \text{Corr}(1_{\{\tau_i \le t\}}, 1_{\{\tau_j \le t\}})$ between two obligors $i \neq j$, as a function of time t. Especially, if we can study several pairs for a fixed obligor i, simultaneously. Recall that we have calibrated $\rho_{i,j}(T)$ against $0.05\rho_{i,j}^{(\text{CDS})}(T)$ for $T = 5$, as discussed in Subsection 4.2.

We first consider the same example as in the previous subsection, where the CDS spreads for Continental is positively correlated with DaimlerChrysler, Peugeot, Renault and Valeo, and negatively correlated with Volvo, BMW, Michelin, Fiat and Volkswagen. We therefore suspect that for most time points t, the corresponding default correlations $\rho_{\text{Cont},j}(t)$ are positive for $j = \text{Volv}, \text{BMW} \dots, \text{CW}$ and negative for $j = \text{DCX}, \text{Peu}, \dots, \text{Valeo}$. This is confirmed by Figure 11. Note that the correlations have parabolic shapes as function of time t.

Furthermore, the CDS-correlation matrix in Table 4 indicate a strong positive correlation among the different CDS-spreads for the banks. This is also the case for the corresponding default correlations, as seen in Figure 12, which displays the correlation between Deutsche Bank and the other banks.

5.4. **The implied expected default times and their ordering.** In this subsection we study implied expected default times $\mathbb{E}[\tau_i]$ and the implied expected ordered default times $\mathbb{E}[T_k]$ for the two calibrated portfolios in Subsection 5.1.

If we order the sequence $\{\mathbb{E}[\tau_i]\}$ in increasing order $\{\mathbb{E}[\tau_{i_k}]\}$ so that $\mathbb{E}[\tau_{i_k}] < \mathbb{E}\left[\tau_{i_{k+1}}\right]$ and study the corresponding sequence of model CDS-spreads $\{R_{i_k}\}$, one would expect that $\{R_{i_k}\}$ are strictly decreasing. However, from Table 8 and Table 9 we see that this is far from true.

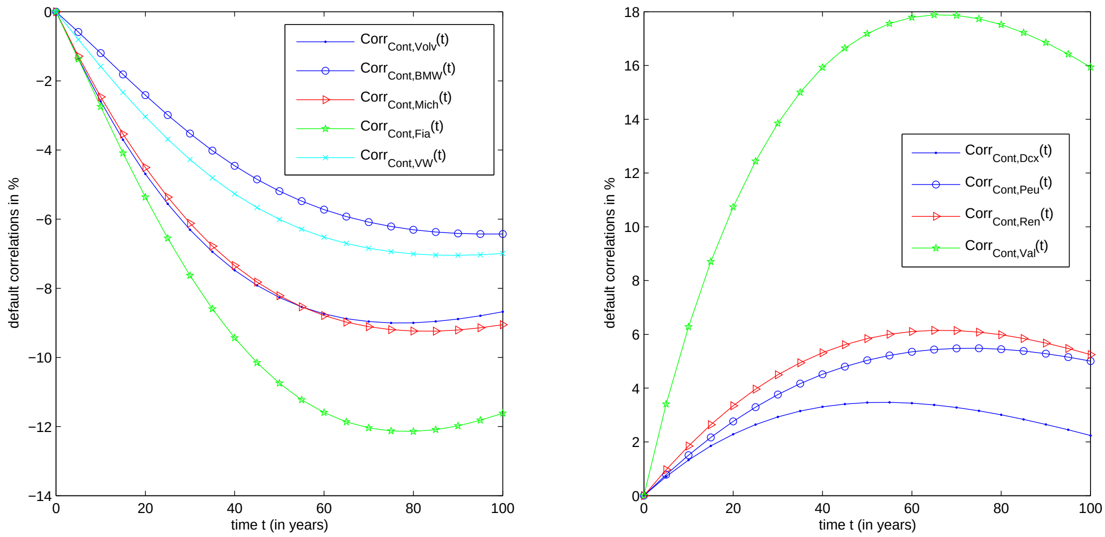

Figure 11: The default correlations between Continental and the companies in the auto portfolio which are negatively correlated (left) and positively correlated (right) with Continental.

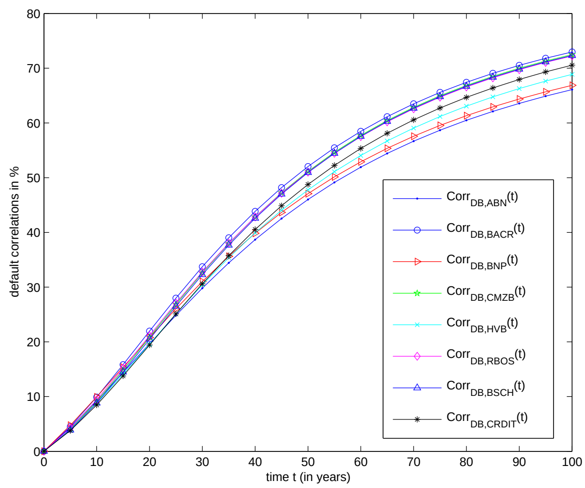

Figure 12: The default correlations between Deutsche Bank and the other banks in the financial portfolio.

Table 8: The expected default times in the auto portfolio, sorted in increasing order $\mathbb{E}[\tau_{i_k}] < \mathbb{E}[\tau_{i_{k+1}}]$, and the corresponding model CDS-spreads.

	VALE	FIAT	DCX	RENA	PEUG	MICH	VOLV	VW	CONT	BMW
$\mathbb{E}[\tau_{i_k}]$	47.3	66.9	68.1	78.9	86.3	86.6	89.2	91.8	116	118
R_{i_k}	66	58	44	28.7	24.9	25.5	25.9	22.1	43.7	9.59

Table 9: The expected default times in the financial portfolio, sorted in increasing order $\mathbb{E}[\tau_{i_k}] < \mathbb{E}[\tau_{i_{k+1}}]$, and the corresponding model CDS-spreads.

	DB	BSCH	CMZB	BACR	CRDIT	RBOS	HSBC	HVB	BNP	ABN
$\mathbb{E}[\tau_{i_k}]$	113	114	116	117	120	120	126	127	131	133
R_{i_k}	13.5	8.22	9.41	6.9	9.99	6.84	7.25	7.22	6.56	6.23

In the financial portfolio, the spreads $\{R_{i_k}\}$ are not decreasing. The auto portfolio has a decreasing trend in the sequence $\{R_{i_k}\}$, except for the Continental spread, $R_{\text{Cont}} = 43.7$ which is the forth biggest spread, while $\mathbb{E}[\tau_{\text{Cont}}] = 116$ years, is the ninth biggest expected default time in the auto portfolio. These irregularities are likely due to the dependence structure in (2.1), (4.2.1), which plays a major roll in the calibration. For example, in the auto case, the CDS spread for Continental is negatively correlated with the spreads for Volvo, BMW, Michelin, Fiat and Volkswagen which means that Continental will benefit from defaults on these firms. In Table 3 we also note that for Continental, the average of the absolute value for the negative correlations is bigger than the corresponding quantity for the positive correlations and no other car company has so many negative default correlations. These observations *may* explain why $\mathbb{E}[\tau_{\text{Cont}}]$ is the ninth biggest in the sequence $\{\mathbb{E}[\tau_{i_k}]\}$. Hence, from an average default timing point of view, Continental is the second less riskiest company in the auto portfolio, even though the CDS spread is the third biggest. Note however that the base intensity a_{Cont} is the third biggest in the auto basket, see Table 13 in Appendix.

These examples indicate that it can be misleading to use the reverse ordering of the CDS spreads as a measure for the relative default riskiness among the obligors in the portfolio.

Table 10: The expected ordered default times $\mathbb{E}[T_k]$.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
Auto	17.8	33.7	50.2	62.9	74.1	85.1	96.8	111	131	186
Financial	85.3	98.7	107	113	118	124	129	136	145	162

Other observations are that the difference between the smallest and biggest expected default time, is 19.5 years in the financial portfolio and 71 years in the auto portfolio.

Also note that in the banking portfolio, the smallest expected default time lie between the expected value of the fourth and fifth ordered default time and the biggest between the seventh and eight. The corresponding quantities in the auto case lie between the second and third, and between the eight and ninth expected ordered default time.

6. Computation of the matrix exponential

All results derived in this paper include computations of the matrix exponential. In this section we describe the method for computing $e^{\boldsymbol{Q}t}$ that is used throughout this article, the so called uniformization method (sometimes also is called the randomization method). It works as follows. Let $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$ and set $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$. Then, $e^{\widetilde{\boldsymbol{P}}\Lambda t} = e^{\boldsymbol{Q}t}e^{\Lambda t}$ since $\boldsymbol{I}$ commutes with all matrices, and using the definition of the matrix exponential renders

$$e^{\boldsymbol{Q}t} = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}. \tag{6.1}$$

Recall that $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ and define $\widetilde{\boldsymbol{p}}(t,N) = \boldsymbol{\alpha}\sum_{n=0}^{N} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}$. Furthermore, for a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\|x\|_1$ denote the L_1 norm, that is $\|x\|_1 = \sum_{i=1}^n |x_i|$. Given $\boldsymbol{Q}$, the uniformization method allows us to find the L_1 approximation error for $\widetilde{\boldsymbol{p}}(t,N)$ apriori, as shown in the following lemma, stated in e.g. [16] and [27], but without a proof.

Lemma 6.1. *Let $\varepsilon > 0$ and pick $N(\varepsilon)$ so that $1 - \sum_{n=0}^{N(\varepsilon)} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \varepsilon$. Then,*

$$\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1 < \varepsilon. \tag{6.2}$$

Proof. By construction, all elements in $\widetilde{\boldsymbol{P}}$ are in $[0,1]$ and all rows in $\widetilde{\boldsymbol{P}}$ sums up to one. We can therefore view $\widetilde{\boldsymbol{P}}$ as a transition matrix for a discrete time Markov chain on $\boldsymbol{E}$. Since $\boldsymbol{\alpha}$ is a probability distribution on $\boldsymbol{E}$ we conclude that $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 = 1$ for all $n \in \mathbb{N}$. These observations imply

$$\begin{aligned}
\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1 &= \Big\| \sum_{n=N(\varepsilon)+1}^{\infty} \boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} \Big\|_1 \\
&\leq \sum_{n=N(\varepsilon)+1}^{\infty} \|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} \\
&= 1 - \sum_{n=0}^{N(\varepsilon)} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \varepsilon
\end{aligned} \tag{6.3}$$

which proves the lemma. □

The lemma implies that, given $\boldsymbol{Q}$, we can for any $\varepsilon > 0$ find a $N(\varepsilon)$ so that $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ approximates $\boldsymbol{p}(t)$ with an accumulated absolute error which is less than ε. Note that the sharp error estimation in Lemma 6.1 relies on a probabilistic argument leading to $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 = 1$ for all $n \in \mathbb{N}$. It is tempting to try to prove (6.2) without this observation, by using that $\|\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n\|_1 \leq \|\boldsymbol{\alpha}\|_1\|\widetilde{\boldsymbol{P}}\|_1^n = \|\widetilde{\boldsymbol{P}}\|_1^n$ where $\|\widetilde{\boldsymbol{P}}\|_1$ is the corresponding matrix norm, and then try show that $\|\widetilde{\boldsymbol{P}}\|_1$ is smaller or equal to one. However, it is easy to see that $\|\widetilde{\boldsymbol{P}}\|_1 > 1$ for $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$ when $\boldsymbol{Q}$ is the generator of a transient Markov process on a finite state $\boldsymbol{E}$ with a final absorbing state, where $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$. This implies that if we use the uniformization method for an arbitrary matrix $\boldsymbol{Q}$, which is not a generator, it may be difficult to find effective apriori error estimates. For such matrices the elements in $\widetilde{\boldsymbol{P}}$ may not even be positive, which makes this method no better than the standard Taylor-series expansion method.

The probabilistic argument for the matrix $\widetilde{\boldsymbol{P}}$ in Lemma 6.1 is no coincidence. The following result can be found in [20].

Theorem 6.2. *Let $(Y_t)_{t\geq 0}$ be a Markov jump process on a finite state $\boldsymbol{E}$ with generator $\boldsymbol{Q}$ where $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\} < \infty$. Then there exists a discrete time Markov chain $(X_n)_{n=0}^{\infty}$ on $\boldsymbol{E}$ with transition matrix $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$ and a Poisson process N_t with intensity Λ, independent of $(X_n)_{n=0}^{\infty}$, such that the processes $(X_{N_t})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have the same finite dimensional distributions.*

Recall that the $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j}\in\boldsymbol{E}}$ so $\boldsymbol{p}_{\boldsymbol{j}}(t) = \mathbb{P}[Y_t = \boldsymbol{j}]$ and Theorem 6.2 implies that

$$
\begin{aligned}
\boldsymbol{p}_{\boldsymbol{j}}(t) &= \mathbb{P}[Y_t = \boldsymbol{j}] \\
&= \mathbb{P}[X_{N_t} = \boldsymbol{j}] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[X_{N_t} = \boldsymbol{j} \mid N_t = n]\, \mathbb{P}[N_t = n] \\
&= \sum_{n=0}^{\infty} \mathbb{P}[X_n = \boldsymbol{j}]\, e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}.
\end{aligned}
\tag{6.4}
$$

Define the row vectors $\boldsymbol{\varphi}(n) = \left(\boldsymbol{\varphi}_{\boldsymbol{j}}(n)\right)_{\boldsymbol{j}\in\boldsymbol{E}}$ for $n \in \mathbb{N}$ as $\boldsymbol{\varphi}_{\boldsymbol{j}}(n) = \mathbb{P}[X_n = \boldsymbol{j}]$ when $n \in \mathbb{N}\setminus\{0\}$ and $\boldsymbol{\varphi}(0) = \boldsymbol{\alpha}$. Theorem 6.2 then implies that $\boldsymbol{\varphi}(n) = \boldsymbol{\varphi}(n-1)\widetilde{\boldsymbol{P}}$ which together with Equation (6.4) renders

$$
\boldsymbol{p}(t) = \sum_{n=0}^{\infty} \boldsymbol{\alpha} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}. \tag{6.5}
$$

But we also know that $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$, so (6.5) is therefore Equation (6.1) restated.

Further benefits with the uniformization method is that all entries in $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ are positive so there are no cancelation effects and the approximation error decreases monotonically with increasing N. If we set $f(t,N) = 1 - \sum_{n=0}^{N} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}$ then $\frac{\partial f(t,N)}{\partial t} = e^{-\Lambda t}\frac{(\Lambda t)^N}{N!} > 0$ so for a fixed N, the approximation error is bounded by a strictly increasing function in t. This is practical, since we then only have to compute one error tolerance for T, that will uniformly bound the error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t,N)\|_1$ for all $t \leq T$. For example, when approximating $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} e^{-rt_n}$ where $t_1 < \ldots < t_{n_T} = T$, we choose $N(\varepsilon/n_T)$ so $1 - \sum_{n=0}^{N(\varepsilon/n_T)} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} < \frac{\varepsilon}{n_T}$ which implies that the total approximation error for the sum $\sum_{n=1}^{n_T} \boldsymbol{\alpha} e^{\boldsymbol{Q}t_n} e^{-rt_n}$ is smaller than ε (since $e^{-rt_n} \leq 1$ for every n).

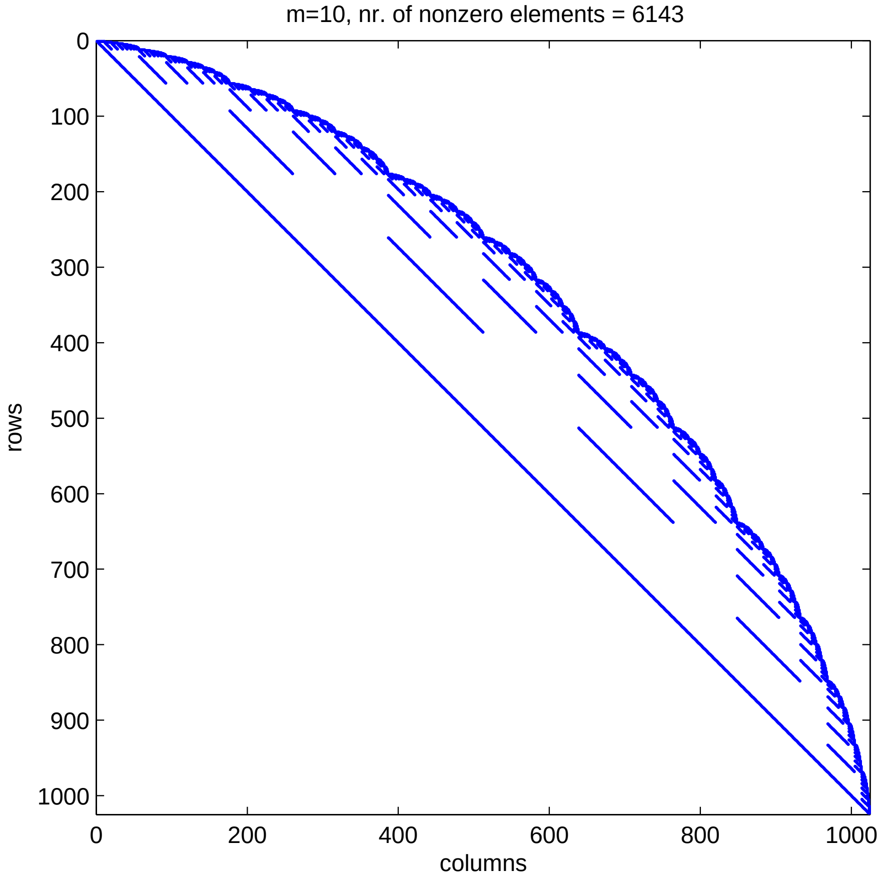

Figure 13: The structure of the nonzero elements in the sparse matrix $\boldsymbol{Q}$ where $m = 10$.

A further point is that our matrices in general are very large, for example if $m = 10$ then the generator has $2^{10} = 1024$ rows and thus contain $2^{20} \approx 1.$ millon entries. However, at the same time it is extremely sparse, see Figure 13. For $m = 10$ there are only 0.59% nonzero entries in $\boldsymbol{Q}$, and hence only about 6100 elements have to be stored, which roughly is the same as storing a full quadratic matrix with 78 rows.

A final point is that we are not interested in finding the matrix exponential itself, but only the probability vector $\boldsymbol{p}(t)$, or a subvector of $\boldsymbol{p}(t)$. This is important, since computing $e^{\boldsymbol{Q}t}$ is very time and memory consuming compared with computing $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$.

For more on the uniformization method with applications in credit derivatives valuations and credit risk, see e.g. [19] and [23].

7. Discussion and conclusions

In this paper we considered the intensity based default contagion model (2.1), where the default intensity of one firm is allowed to change when other firms default. The model was reinterpreted in terms of a Markov jump process, a so called multivariate phase-type distribution. This reinterpretation made it possible to derive practical formulas for many quantities, such as multivariate default and survival distributions, conditional multivariate distributions, marginal distributions, multivariate densities, correlations, expected default times, CDS-spreads and so on.

In the model we used two CDS portfolios for numerical studies, one in the European auto sector, the other in the European financial sector. Both baskets contained 10 companies. For an exogenously given dependence matrices $\{D_{i,j}\}$, we calibrated the portfolios against their market CDS spreads and the corresponding CDS-correlations. In both portfolios the CDS-fits where perfect, and in the financial case the correlation fit was also perfect, while the autos correlation matching was mediocre.

We then computed the implied joint default and survival distributions, the implied univariate and bivariate conditional survival distributions, the implied default correlations, and the implied expected default times and expected ordered defaults times. Qualitatively, many of the results where as expected. However it would seem rather impossible to guess the sizes of the probabilities and other quantities, without computation.

Future extensions of the model (2.1) is for example to include first-to-default swaps, other portfolio credit derivatives and corporate information, so that $\{D_{i,j}\}$ can be determined more realistically. Further empirical investigations of the approximation $\rho_{i,j}(T) \sim \beta\rho_{i,j}^{(\mathrm{CDS})}(T)$ is also needed.

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8. Appendix

The dependence matrix $\{D_{i,j}\}$ for the financial portfolio was generated by drawing a random matrix where the entities lie in the interval $1, 2, \ldots, 45$. If some of the elements in $1, 2, \ldots, 45$ are not present in the sampling, we removed doublets in $D_{i,j}$ until all integers between 1 and 45 were present.

The $\{D_{i,j}\}$ matrix for the auto portfolio was created in the same way. However, we also made sure that $\{D_{i,j}\}$ was constructed so that it split $\{d_q\}$ in two disjoint groups, $\{d_q\} = \boldsymbol{d}_- \cup \boldsymbol{d}_+$ such that if $\rho_{i,j}^{(\text{CDS}}(T)) < 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_-$ and if $\rho_{i,j}^{(\text{CDS})}(T) \geq 0$ then $d_{D_{i,j}} \in \boldsymbol{d}_+$, where $\rho_{i,j}^{(\text{CDS})}(T))$ is the CDS-correlation matrix, retrieved from market data. Furthermore, we also constructed $\boldsymbol{d}_-$ so that $d_p \in \boldsymbol{d}_-$ only appeared once in the matrix

$\{D_{i,j}\}$. Then we constrained the parameters in $\boldsymbol{d}_-$ as follows

$$d_p \leq \frac{1}{|N_i|} - 0.005 \quad \text{if} \quad \theta_{i,j} = -d_p \quad \text{for some} \quad j \in N_i,$$

where N_i is the set of obligors $j \neq i$ which are negatively correlated with entity i, that is, where $\rho_{i,j}^{(\mathrm{CDS})}(T) < 0$.

Table 11: The dependence matrix $D_{i,j}$ for the autos portfolio. Entries with a negative subscript indicates that the corresponding entry in the correlation matrix is negative.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	0	28	45	5_-	29	25	10	26	36	2
BMW	10	0	16	34_-	35	31	40	7	43_-	1
MICH	27	41	0	20_-	31	7	37	29	39	40
CONT	22_-	6_-	4_-	0	17	24_-	29	26	40	9_-
DCX	40	15	2	21	0	7	37	17	8_-	13
FIAT	45	42	25	19_-	23	0	23	32	44_-	23
PEUG	21	42	25	29	32	17	0	25	12	13
RENA	2	41	27	39	35	39	13	0	11	21
VALE	37	18_-	12	25	14_-	38_-	15	31	0	3_-
VW	16	40	25	30_-	25	27	39	28	33_-	0

Table 12: The dependence matrix $D_{i,j}$ for the financial portfolio.

	ABN	BACR	BNP	CMZB	DB	HSBC	HVB	RBOS	BSCH	CRDIT
ABN	0	10	31	39	22	28	15	20	1	2
BACR	13	0	34	4	29	17	35	12	35	41
BNP	8	6	0	22	2	15	35	19	21	44
CMZB	41	24	24	0	15	40	5	31	2	30
DB	12	31	15	21	0	11	15	38	4	33
HSBC	17	32	3	27	4	0	12	37	43	22
HVB	29	42	12	9	31	14	0	18	27	31
RBOS	14	1	45	23	1	8	40	0	11	16
BSCH	9	26	17	7	14	32	35	7	0	18
CRDIT	10	25	35	33	2	36	11	15	19	0

Table 13: The calibrated base intensities.

	a_{VOL}	a_{BMW}	a_{MICH}	a_{CON}	a_{DCX}	a_{FIA}	a_{PEU}	a_{REN}	a_{VALE}	a_{VW}	
Auto	34.7	16.5	43.2	65.4	71.5	94.4	31.5	43.3	128	33.7	$\times 10^{-4}$
Financial	7.85	10.5	10.5	11.4	20.7	10.6	8.48	8.70	13.8	14.8	$\times 10^{-4}$

Table 14: The dependence variables d_q s.t $\theta_{i,j} = \pm d_{D_{i,j}}$ for the autos portfolio. Entries with a negative subscript indicates that $\theta_{i,j} = -d_{D_{i,j}}$.

$d_1, \ldots, d_9$	5.49	0.903	0.245_-	0.195_-	0.995_-	0.195_-	0.675	0.0191_-	0.195_-
$d_{10}, \ldots, d_{18}$	3.93	0	0.434	1.78	0.245_-	0.946	0.383	0.445	0.245_-
$d_{19}, \ldots, d_{27}$	0.495_-	0.813_-	0	0.195_-	0.394	0.195_-	1.62	0.758	2.83
$d_{28}, \ldots, d_{36}$	1.53	0.355	0.495_-	1.11	0.826	0.495_-	0.495_-	1.17	0
$d_{37}, \ldots, d_{45}$	1	0.142_-	0	0	2.66	3.53	0.495_-	0.495_-	0.936

Table 15: The dependence variables d_q s.t $\theta_{i,j} = d_{D_{i,j}}$ for the financial portfolio.

$d_1, \ldots, d_9$	7.66	4.56	5.43	1.63	9.52	0	7.15	17.9	6.76
$d_{10}, \ldots, d_{18}$	8.68	9.39	10.4	9.68	6.18	7.61	5.2	12.2	11
$d_{19}, \ldots, d_{27}$	1.45	14.3	1.47	2.42	13.4	13.3	0	6.4	7.48
$d_{28}, \ldots, d_{36}$	0.482	7.88	9.77	3.07	4.72	2.98	17	7.34	9.32
$d_{37}, \ldots, d_{45}$	1.47	3.37	4.83	7.21	13.3	9.7	8.22	4.76	17.5

Table 16: The absolute calibration errors for the default correlation matrices, in percent of matrix $\{0.05\rho_{i,j}^{(\text{CDS})}(T)\}$, for the auto portfolio.

	VOLV	BMW	MICH	CONT	DCX	FIAT	PEUG	RENA	VALE	VW
VOLV	0									
BMW	3.4	0								
MICH	9.6	32	0							
CONT	46	83	12	0						
DCX	12	17	9.1	9.4	0					
FIAT	8.9	7.9	2.8	57	11	0				
PEUG	2	49	39	11	31	27	0			
RENA	35	89	14	93	9.1	20	23	0		
VALE	45	64	19	55	7.3	4.7	9.1	0.41	0	
VW	120	0.35	26	20	5.1	24	97	6.8	47	0

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